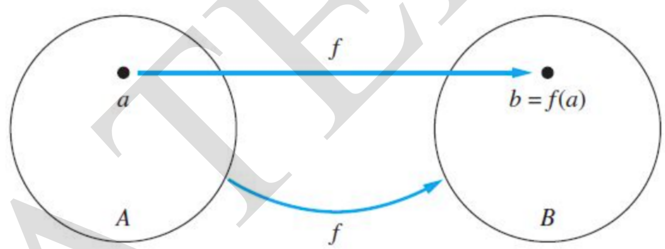
# Chapter 3

# Functions

Let $A$ and $B$ be nonempty sets. A function $f$ from $A$ to $B$ is an assignment of exactly one element of $B$ to each element of $A$. We write $f(a) = b$ if $b$ is the unique element of $B$ assigned by function $f$ to the element $a$ of $A$. If $f$ is a function from $A$ to $B$, we write $f : A \to B$.

If $f$ is a function from $A$ to $B$, we say that $A$ is the *domain* of $f$ and $B$ is the *codomain* of $f$. If $f(a) = b$, we say that $b$ is the *image* of $a$ and $a$ is a *preimage* of $b$. The *range*, or *image*, of $f$ is the set of all images of elements of $A$. Also, if $f$ is a function from $AtoB$, we say that $f$ maps $A$ to $B$



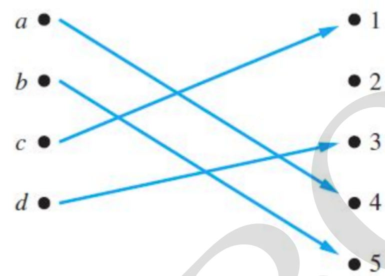

**The Function *f* Maps *A* to *B*.**

Let $f$ be the function that assigns the last two bits of a bit string of length 2 or greater to that string. For example, $f(11010) = 10$. Then, the domain of $f$ is the set of all bit strings of length 2 or greater, and both the codomain and range are the set $\{00, 01, 10, 11\}$.

Let $f_1$ and $f_2$ be functions from $A$ to $\mathbf{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from $A$ to $\mathbf{R}$ defined for all $x \in A$ by

$(f_1 + f_2)(x) = f_1(x) + f_2(x)$
$(f_1 f_2)(x) = f_1(x) f_2(x)$

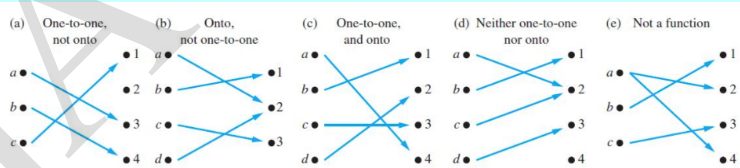
Let $f$ be a function from $A$ to $B$ and let $S$ be a subset of $A$. The *image* of S under the function $f$ is the subset of $B$ that consists of the images of the elements of $S$. We denote the image of $S$ by $f(S)$, so
$f(S) = \{t | \exists s \in S(t = f(s))\}$.
We also use the shorthand $\{f(s) | s \in S\}$ to denote this set.




**A One-to-One Function.**

A function $f$ whose domain and codomain are subsets of the set of real numbers is called *increasing* if $f(x) \leq f(y)$, and strictly increasing if $f(x) < f(y)$. whenever $x < y$ and $x$ and $y$ are in the domain of $f$. Similarly, $f$ is called decreasing if $f(x) \geq f(y)$, and strictly decreasing if $f(x) > f(y)$, whenever $x < y$ and $x$ and $y$ are in the domain of $f$. (The word strictly in this definition indicates a strict inequality.)

A function $f$ from $A$ to $B$ is called onto, or a surjection, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function $f$ is called surjective if it is onto.




The function $f$ is a one-to-one correspondence, or a bijection, if is both one-to-one and onto. We also say that such a function is bijective.

Let $A$ be a set. The identity function on $A$ is the function $\iota_A : A \to A$, where

$$\iota_A(x) = x$$

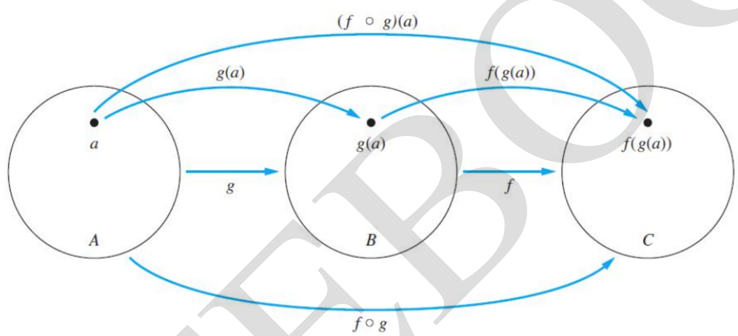
for all $x \in A$. In other words, the identity function $\iota_A$ is the function that assigns each element to itself. The function $\iota_A$ is one-to-one and onto, so it is a bijection. (Note that $\iota$ is the Greek letter iota.)

Let $f$ be one-to-one correspondence from the set $A$ to the set $B$. The inverse function of $f$ is the function that assigns to an element $b$ belonging to $B$ the unique element $a$ in $A$ such that $f(a) = b$. The inverse function of $f$ is denoted by $f^{-1}$. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

A one-to-one correspondence is called **invertible** because we can define an inverse of this function. A function is **not invertible** if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

Let $g$ be a function from the set $A$ to the set $B$ and let $f$ be a function from the set $B$ to the set $C$. The composition of the functions $f$ and $g$, denoted for all $a \in A$ by $f \circ g$, is defined by

$(f \circ g)(a) = f(g(a))$.




| **Useful Properties of the floor and Ceiling Functions.** (n is an integer, x is a real number) | |
|---|---|
| (1$a$) | $\lfloor x \rfloor = n$ if and only if $n \le x < n+1$ |
| (1$b$) | $\lceil x \rceil = n$ if and only if $n-1 < x \le n$ |
| (1$c$) | $\lfloor x \rfloor = n$ if and only if $x-1 < n \le x$ |
| (1$d$) | $\lceil x \rceil = n$ if and only if $x \le n < x+1$ |
| (2) | $x-1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x+1$ |
| (3$a$) | $\lfloor -x \rfloor = -\lceil x \rceil$ |
| (3$b$) | $\lceil -x \rceil = -\lfloor x \rfloor$ |
| (4$a$) | $\lfloor x+n \rfloor = \lfloor x \rfloor + n$ |
| (4$b$) | $\lceil x+n \rceil = \lceil x \rceil + n$ |