Chapter 7

Group Theory

1. In the following determine whether the systems described are groups. If they are not, point out which of the group axioms fail to hold.

   (a) G = set of all integers, $a.b \equiv a - b$

   (b) G = set of all positive integers, a.b = ab, the usual product of integers.

   (c) $G = a_0, a_1, \ldots, a_6$ where $a_i.a_j = a_{i+j}$ if $i + j < 7$, $a_i.a_j = a_{i+j-7}$ if $i + j \geq 7$, (for instance, $a_5.a_4 = a_{5+4-7} = a_2$ since $5 + 4 = 9 > 7$)

   (d) G set of all rational numbers with odd denominators, $a.b \equiv a - b$, the usual addition of rational numbers.

2. Prove that if G is an abelian group, then for all $a, b \in G$ and all integers n $(a.b)^n = a^n.b^n$

3. If G is a group such that $(a.b)^2 = a^2.b^2$ for all $a, b \in G$, show that G must be abelian.

4. If G is a finite group, show that there exists a positive integer N such that $a^N = e$ for all $a \in G$.

5. Show :

   (a) If the group G has three elements, show it must be abelian.

   (b) Do part (a) if G has four elements.

   (c) Do part (a) if G has five elements.

6. Show that if every element of the group G is its own inverse, then G is abelian.

7. If G is a group of even order, prove it has an element $a \neq e$ satisfying $a^2 = e$.

8. Suppose a finite set G is closed under an associative product and that both cancellation laws hold in G. Prove that G must be a group.
9. If H and K are subgroups of G, show that $H \cap K$ is a subgroup of G.
10. Prove that if G is a group and $x^2 = 1$ for all $x \in G$, then G is abelian.