Chapter 8

Graph Theory

1. Suppose that $d_1, d_2, \ldots, d_n$ is a graphic sequence. Show that there is a simple graph with vertices $v_1, v_2, \ldots, v_n$ such that $deg(v_i) = d_i$ for $i = 1, 2, \ldots, n$ and v_i is adjacent to $v_2, \ldots, v_{d_i+1}$

2. Show that a sequence $d_1, d_2, \ldots, d_n$ of non negative in non increasing order is a graphic sequence if and only if the sequence obtained by reordering the terms of the sequence $d_2 - 1, ..., d_{d_1+1} - 1, d_{d_2+2}, ..., d_n$ so that the terms are in non increasing order is a graphic sequence.

3. Show that if G is a bipartite simple graph with v vertices and e edges, then $e \leq v^2/4$.

4. Describe an algorithm to decide whether a graph is bipartite based on the fact that a graph is bipartite if and only if it is possible to color its vertices two different colors so that no two vertices of the same color are adjacent.

5. Show that if a bipartite graph $G = (V, E)$ is n-regular for some positive integer n and (V_1, V_2) is a bi-partition of , then .That is, show that the two sets in a bi partition of the vertex set of an n-regular graph must contain the same number of vertices .

6. Let G be a graph with adjacency matrix A with respect to the ordering $v_1, v_2, \ldots, v_n$ of the vertices of the graph (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length r from v_i to v_j , where r is a positive integer, equals the $(i, j)^t h$entry of A^r.

7. Show that in every simple graph there is a path from every vertex of the odd degree to some other vertex of odd degree.

8. Suppose that v is an endpoint of a cut edge. Prove that v is a cut vertex if and only if this vertex is not pendant.

9. Show that a vertex c in the connected simple graph G is a cut vertex if and only if there are vertices u and v, both different from c, such that every path between u and v passes through c.

10. Show that a simple graph with at least two vertices has at least two vertices that are not cut vertices.

11. Show that an edge in a simple graph is a cut edge if and only if this edge is not part of any simple circuit in the graph.

12. Show that if a simple graph G has k connected components and these components have $n_1, n_2, ..., n_k$ vertices, respectively, then the number of edges of G does no exceed

$$\sum_{i=1}^{k} C(n_i, 2)$$

13. Show that a simple graph G is bipartite if and only if it has no circuits with an odd number of edges.

14. Suppose that G is an simple graph with n vertices, $n \geq 3$ and $deg(x) + deg(y) \geq n$ whenever x and y are nonadjacent vertices in G. Ore's theorem states that under these conditions, G has a Hamilton circuit.

    An orientation of an undirected simple graph is an assignment of directions to its edges such that the resulting directed graph is strongly connected. When the orientation of an undirected graph exists, this graph is called orientable.

15. Show that a graph is not orientable if it has a cut edge.

    A rooted tree is a tree in which one vertex has been designated as the root and every edge is directed away from the root.

16. Show that a full m-ary balanced tree of height h has more than $m^h - 1$ leaves

17. Suppose that $d_1, d_2, \ldots, d_n$ are n positive integers with sum 2n - 2. Show that there is a tree that has n vertices such that the degrees of these vertices are $d_1, d_2, \ldots, d_n$.