Part VI

Answers and Discussion Functions

Part VII

Answers and Discussion Group Theory

1. $(a.b)^n = a^n.b^n \quad \forall a, b, \in G$ and $\forall n$
   we can prove it using Induction.
   when $n = 0$
   $(a.b)^0 = a^0.b^0$
   $e = e.e$
   $e = e$
   Assume that when $n = k$ it is true.
   $(a.b)^k = a^k.b^k$
   We need to prove it for $k + 1$
   $(a.b)^{k+1} = a^{k+1}b^{k+1}$
   $(a.b)(a.b)^k = a^{k+1}b^{k+1}$ [Induction Hypothesis]
   $(a.b)(b^k.a^k) = a^{k+1}b^{k+1}$ [Commutative Law]
   $a.(b.b^k).a^k = a^{k+1}b^{k+1}$ [Associative Law]
   $a.b^{k+1}.a^k = a^{k+1}b^{k+1}$
   $(a.a^k).b^{k+1} = a^{k+1}b^{k+1}$ [Commutative Law]
   $a^{k+1}.b^{k+1} = a^{k+1}b^{k+1}$ [Associative Law]

2. $(a.b)^2 = a^2.b^2 \forall a, b \in G$
   $(a.b)(a.b) = a^2.b^2$
   $a.b.a.b = a.a.b.b$ [Associative]
   $b.a.b = a.b.b$ [Left Cancellation]
   $b.a = a.b$ [Right Cancellation]
   $\forall a, b \in G \quad b.a = a.b$
   Hence G is abelian.

3. G is a finite group, $a \in G$
   Observe $a^0, a^1, ..., a^n$. [Where n is the order of the group]
   Since there are $n + 1$ elements at least 2 should be same.
   $a^i = a^j, i > j$
   $a^{i-j} = e$ [Left cancellation j times]
   $i - j > 0$ as $i > j$
   Now we have $N = i - j > 0$ such that $a^N = e$ for all $a \in G$

4. Since there are 3 elements, say they are e, a, b
   $e * a = a * e = a$ [Def. of identity element]
   $e * b = a * b = b$ [Def. of identity element]
   $a * b$ can not be equal to a [Why?]
   $\not{a} * b = \not{a}$ then $b = e$ [e & b are distinct elements]
   Similarly $a * b$ can not be equal to b
   now only possibility is $a * b = e$
   Same like above we can prove that $b * a = e$
   $a * b = b * a = e$
   We have proved that $\forall x, y \in G$
   $x * y = y * x$

5. It is given that $a^{-1} = a \quad \forall a \in G$
   $a * a^{-1} = e$

$a * a = e[a^{-1} = a] \longrightarrow (1)$
take an element $a * b$
$(a * b) * (a * b) = e$ [from (1)]
$a * b * a * b = e$ [Associativity]
$b * a * b = a^{-1} * e$ [Apply a^{-1}]
$b * a * b = a^{-1}$
$a * b = b^{-1} * a^{-1}$ [Apply b^{-1}]
$a * b = b * a$ [b^{-1} & $a^{-1} = a$]
We proved that $a * b = b * a \quad \forall a, b \in G$
Hence G is abelian.

6. G is a group of even order, assume it as $2n$.
   e is it's own inverse.
   Other than e there are $2n - 1$ elements in G.
   For example a is an element which is not it's own inverse
   then $\exists b \in G, b \neq a$ such that $b * a = a * b = e$
   b can not be inverse of any other element (why?)
   $c \neq a$ and $b * c = c * b = e$
   $b * a = e, b * c = e$
   $\not{b} * a = \not{b} * c$
   $a = c$ [contradiction]
   a and b are inverse to each other. we can pair up all the element x, y which are inverse to each other.
   Since there are odd number of elements, at least
   one element can not be paired. so for that element x_1
   $x * x = e$

7. **Given:** G is closed, Associative, finite
   Both cancellation laws hold good.
   $a * b = a * c$ then $b = c$ [Left cancellation]
   $b * c = a * c$ then $b = a$ [Right cancellation]
   Let us apply an operation $a * x \quad \forall x \in G$
   $f(x) = a * x$ defined for every $x \in G$
   Since G is closed $a * x \in G$
   $f(x)$ is a function from $G \to G$

   f is one-one (why?)
   If not, $\exists x, y \in G, x \neq y$
   $f(x) = f(y) \;\; \not{a} * x = \not{a} * y$
   $x = y$!contradiction
   f is onto (why?)
   Any one-one function from finite set A to A is alwas onto.
   Since f is onto there is a mapping f such that $f(x) = a$ for some x
   $a * x = a \longrightarrow (1)$
   Let b be an arbitrary element of G.
   $a * x * b = a * b$

$x * b = b \longrightarrow (2)$

$b * x * b = b * b$ [Applied $b*$]
$x * b = b \longrightarrow (3)$

From (2) and (3) $x*b = b*x$ for every element $b \in G$ hence x is an identity element.
Now we need to show inverse.
Since f is an onto function, $\exists y \in G$ such that
$f(y) = e$
$a * y = e \longrightarrow (4)$
$a * y * a = e * a$
$\not{a} * y * a = \not{a} * e$ [e is an identity element]
$y * a = e \longrightarrow (5)$ [Left cancellation]

from (4) (5) $a * y = y * a = e$
same thing can be for any element a.
C for example to show inverse for b, take new function $f(x) = b * x$

8. H and K are Subgroups of G.
   $H \cap K$

   (a) $\forall a, b \in H \cap K$
       Since $a, b \in H$, $a, b \in K$
       $a * b \in H$ [H is a group]
       $a * b \in K$ [K is a group]
       $a * b \in H \cap K$
       $H \cap K$ is closed.
   (b) $\forall a, b, c \in H \cap K$
       $a, b, c \in H \quad a, b, c \in K$
       $(a * b) * c = a * (b * c)$ [H is a group, so associative]
       $H \cap K$ is associative.
   (c) Since H and K are subgroups of G
       identity element $e \in H, e \in K$
       $e \in H \cap K$.
   (d) $\forall a \in H \cap K$
       Since H is a group $a^{-1} \in H$
       Since K is a group $a^{-1} \in K$
       $a^{-1} \in H \cap K$

   Thus $H \cap K$ is a group