❖ Proposition:

Proposition is a statement which is either true or false but not both at a time.

Example: 1. 2×2=5

2. 3+3=6

3. Mr. Manmohan Singh is pm of India.

❖ Connective:
It is used to connect one or more propositions.

The basic connectives are

1 .Negation

2. Conjunction

3. Disjunction

4. Implication

5. Bi implication

❖ Truth tables:
Truth table is a collection of truth values of a compound proposition whose value is derived from simple propositions, connectives of that of compound proposition.

➢ Truth table of ~P:

P	~P
T	F
F	T

Truth value of ~P is exactly opposite to truth value of P.

➢ Truth table of P ∧ Q:

P	Q	P∧Q
T	T	T
T	F	F

F	T	F
F	F	F

P∧Q is true if and only if P=T Q=T.

- Truth table of P ∨Q:

P	Q	P ∨Q
T	T	T
T	F	T
F	T	T
F	F	F

It is false if and only if P = F Q = F.

- Truth table of P → Q:

It is read **as if P then Q** and also **Q whenever P.**

P	Q	P→ Q
T	T	T
T	F	F
F	T	T
F	F	F

It is false if and only if P = T Q = F.

Example: If 2×2 = 5 then 3×3 = 10

It is nothing but P → Q where P = 2×2 = 5 and Q = 3×3 = 10

Truth value of P = F.

Truth value of Q = T.

Truth value of P → Q is F → T ⇔ T.

- Truth table of "P **bi implication** Q" (P↔Q):

It can be read as P if and only if Q or P iff Q

P	Q	P↔Q
T	T	T
T	F	F
F	T	F
F	F	T

`Example: Delhi is capital of America

iff

Newyork is capital of India.

Here P: Delhi is capital of America (F).

Q: Newyork is capital of India (F).

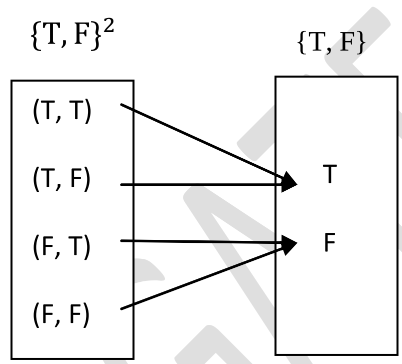
Truth value of (P ⇔ Q) ⇔ (F ⇔ F) ⇔ T.

- **Propositional function:** $f_{\{T,F\}^n \rightarrow \{T,F\}}$ is called propositional function over n variables from $\{T, F\}^n \rightarrow \{T, F\}$.

  An example of propositional function on 2 variables.

  $f_{\{T,F\}^2 \rightarrow \{T,F\}}$

  $\{T, F\}^2 = \{(T,T),(T,F),(F,T),(F,F)\} = \{T,F\} \times \{T,F\}$



- **Propositional Formula:**

  A formula can be recursively defined as follows

  1. T, F are formulae
  2. Any simple proposition is a formula.

     Example: P, Q, R…….

  3. If F_1, F_2 are formulae then $F_1 \wedge F_2$, $F_1 \vee F_2$, $\sim F_1$ are also formulae.

  P, P ∧ P, P ∨P are different formulae whose corresponding propositional functions are same.

  A formula can be classified into three ways

  1. **Tautology:** A propositional formula is said to be Tautology iff it is true in all the cases.

     It is denoted by (T).

Example: P → P

Truth table of p → p

P	P	P →P
T	T	T
F	F	T

2. Contradiction: Propositional formula is called contradiction iff it is false in all the cases.

   It is denoted by (F).

   Example: P∧ ~P

P	~P	P∧ ~P
T	F	F
F	T	F

3. Contingency: A propositional formula is called contingency iff it is neither Tautology nor contradiction.

➢ Priorities of the operators:

∧ has greater priority than ∨, →.

∨ has greater priority than →.

Note:- ~ > ∧ > ∨> → >

➢ Associativity:

1. ∧, ∨ are associative. That means P ∧ Q ∧ R ⇔ P ∧ (Q ∧ R) ⇔ (P ∧ Q) ∧R Similarly for ∨ operator.
2. → has right associativity.

That means P→Q→R = P→ (Q→R)

P→Q→R ≠ (P→Q) →R

➢ Equivalence:

Two propositional formulae are said to be equivalent iff F1↔F2 is tautology.
That means both are true or none is true (both are false).
Example: ~ (p ∧ Q) ⇔ ~p ∨ ~Q

P	Q	(P ∧↔Q)	~ (P ∧Q)	~P∨ ~Q	F1 ↔F2
T	T	T	F	F	T
T	F	F	T	T	T
F	T	F	T	T	T

F	F	F	T	T	T

➢ Equivalence Rules:

Following axioms are called equivalence rules, which are helpful in simplifying the formula.

$P \wedge T \Leftrightarrow P$ $P \vee F \Leftrightarrow p$	Identity laws
$P \vee T \Leftrightarrow T$ $P \wedge F \Leftrightarrow P$	Domination laws
$P \vee P \Leftrightarrow P$ $P \wedge P \Leftrightarrow P$	Idempotent laws
$\sim(\sim P) \Leftrightarrow P$	Double negation laws
$(P \vee Q) \Leftrightarrow (Q \vee P)$ $(P \wedge Q) \Leftrightarrow (Q \wedge P)$	Commutative laws
$P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$ $P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$	Associative laws
$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$ $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$	Distributive laws
$\sim(P \wedge Q) \Leftrightarrow \sim p \vee \sim Q$ $\sim(P \vee Q) \Leftrightarrow \sim P \wedge \sim Q$	Demorgan's law

Some other useful equivalence

$P \rightarrow Q \Leftrightarrow \sim p \vee Q$

$P \leftrightarrow Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$

➢ Logical Implication: (⇒)

$P \Rightarrow Q$ "P logically imply Q"

"Q logically follows p"

$P \Rightarrow Q$ iff $P \rightarrow Q$ is a tautology.

Note: Whenever $F1 \Rightarrow F2$ then it cannot be possible to have F1 as true and F2 as false at same time.

➢ Validity :

A formula is said to be valid iff it is true in all the cases.

Example: $P \wedge Q \rightarrow P \vee Q$

P	Q	$P \wedge Q$	$P \vee Q$	$(P \wedge Q) \rightarrow (P \vee Q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

- ## Satisfiability:

A formula is said to be satisfiable iff it is true in at least one case.
Example: $P \vee Q \rightarrow P \wedge Q$

P	Q	$P \vee Q$	$P \wedge Q$	$P \vee Q \rightarrow P \wedge Q$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	T
F	F	F	F	T

Above formula is satisfiable but not valid.

- ## Inference System:

- Formula $P_1 \wedge P_2 \wedge P_3 \ldots\ldots \wedge P_n \rightarrow C$ is called an inference system.
- An inference system is called valid when $P_1 \wedge P_2 \wedge P_3 \ldots\ldots \wedge P_n \rightarrow C$ is tautology.
- $P_1, P_2, P_3 \ldots\ldots P_n$ are called premises and C is conclusion.
- There is no world or Universe where $P_1, P_2, P_3 \ldots\ldots P_n$ are true and C is false at same time if the inference system is valid.
- If an inference system is not valid then it is called invalid and conclusion is called invalid conclusion.
- Inference system $P_1 \wedge P_2 \wedge P_3 \ldots\ldots \wedge P_n \rightarrow C$ can be viewed as

$$\begin{array}{l} P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_n \\ \hline C \\ \hline \end{array}$$

➢ Inference rules:

Each inference rule is valid inference system.

1. p q $\therefore p \wedge q$	$p \wedge q \rightarrow p \wedge q$	Conjunction
2. p $\therefore p \vee q$	$p \rightarrow p \vee q$	Addition
3. $p \wedge q$ $\therefore p$	$p \wedge q \rightarrow p$	Simplification
4. p $p \rightarrow q$ $\therefore p$	$P \wedge (p \rightarrow q) \rightarrow q$	Modus ponens
5. $\sim q$ $p \rightarrow q$ $\therefore \sim p$	$\sim q \wedge (p \rightarrow q) \rightarrow \sim p$	Modus tollens
6. $p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$	Hypothetical syllogism
7. $p \vee q$ $\sim p$ $\therefore q$	$(p \vee q) \wedge \sim p \rightarrow q$	Disjunctive syllogism
8. $p \vee q$ $\sim p \vee r$ $\therefore q \vee r$	$(p \vee q) \wedge (\sim p \vee r) \rightarrow (q \vee r)$	Resolution

❖ Practice Questions and Explanations:

1) The number of propositional functions on n variables?

a) 2^n b) 2^{2^n} c) n^{2^n} d) n^{n^2}

An n variable propositional is a mapping from $\{T, F\}^n \rightarrow \{T, F\}$

p_1 p_2 p_3 ……… p_n	f
T T T ……….. F	
T T T ……….. F	
. . ….. … .... F F F ……….. F	

2^n

Each propositional function is mapping these 2^n rows to {T, F}.so that each row can be mapped to either T or F

Forming propositional function is nothing but mapping each row with 2 options (T or F)

The number of different mappings for 2n rows = 2 X 2 X 2 X …… 2^n times= 2^{2^n}

2) $p \wedge (p \rightarrow q)$ is a

a) tautology b)contradiction c) contingency d)none

Solution: It can be verified using truth table of $p \wedge (p \rightarrow q) \rightarrow q$ but it takes more time.

The clever way of doing this is shown below.

A formula $F_1 \rightarrow F_2$ cannot be tautology if F_1 = T, F_2 = F

Here $p \wedge (p \rightarrow q) \rightarrow q$ can be viewed as $F_1 \rightarrow F_2$ where $F_1 = p \wedge (p \rightarrow q)$,

$F_2 = q$

To prove that $F_1 \rightarrow F_2$ is not tautology fix F_1 = T and F_2 = F

F1: $p \wedge (p \rightarrow q)$ = T

then p = T and $p \rightarrow q$ = T (this is the only possibility, there is no other possibility) $p \rightarrow q$ = T can be done in so many ways but when p = T then

$p \rightarrow q$ = T can be possible only in one way, that is, q= T.

Now we can verify that when F_1 = T then p = T q = T.

Consider the complete formula $F_1 \rightarrow F_2$

$\overline{p \wedge (p \rightarrow q) \rightarrow} q$

But we wanted to make F_2 = F; since F_2 = q, that means q = F

But already we know that q = T that means whenever F_1 = T, F_2 cannot be.

Hence $F_1 \rightarrow F_2$ cannot be false.

Hence $F_1 \rightarrow F_2 \Leftrightarrow p \wedge (p \rightarrow q) \rightarrow q$ is tautology.

3) $p \wedge (p \rightarrow q) \wedge (q \rightarrow r) \rightarrow r$ is a

a) Tautology b) contradiction c) contingency d) none

Solution: let $F_1 = p \wedge (p \rightarrow q) \wedge (q \rightarrow r)$ and $F_2 = r$

To prove that $F_1 \rightarrow F_2$ is not tautology try to assign $F_1 = T, F_2 = F$
$F_2 = r = F$ ---------------------------------(Eq1)
$F_1 = T$ means $p \wedge (p \rightarrow q) \wedge (q \rightarrow r) = T$
Any formula $F3 \wedge f4 \wedge F5 = T$ means $F3 = T$ and $F4 = T$ $F5 = T$
Here $F5 = q \rightarrow r = T$--------------------------(eq2)
$F4 = p \rightarrow q = T$--------------------------(eq3)
$F3 = p = T$ -------------------------------(eq4)
From eq1 and eq2
$r = F$
$q \rightarrow r = T$
$q \rightarrow F = T$
Q should be false that means $q = F$--------------------------(Eq5)
From eq5 and eq3
$p \rightarrow q = T$ and $q = F$
p cannot be true (think why?)
$p = F$ ----------------------------(eq6)
from eq4 and eq6
$p = T$ and $p = F$
this is not possible. Hence we cannot make $F_1 = T$ and $F_2 =$ false at a time.
$F_1 \rightarrow F_2$ is always true
$F_1 \rightarrow F_2 = p \wedge (p \rightarrow q) \wedge (q \rightarrow r)$ is tautology.

4) $p \vee q \rightarrow p \wedge q$ is
a) Tautology b) contradiction c) contingency d) none of the above
Solution: $F_1 = p \vee q$ and $F_2 = p \wedge q$
It is possible to get $F_1 \rightarrow F_2$ as false. So it cannot be tautology
$F_1 \rightarrow F_2$ cannot be false when $p = F, q = F$. That means $F_1 \rightarrow F_2$ cannot be contradiction. Hence it is contingency.

5) The proposition $p \wedge (\sim p \vee q)$ is logically equivalent to
a) Tautology
b) logically equivalent to $p \wedge q$
c) logically equivalent to $p \vee q$
d) none

Solution: $p \wedge (\sim p \vee q) \Leftrightarrow (p \wedge \sim p) \vee (p \wedge q)$ (distributive law)

$\Leftrightarrow F \vee (p \wedge q)$ $[p \wedge \sim p \Leftrightarrow F]$

$\Leftrightarrow p \wedge q$

There fore give proposition is logically equivalent to $p \wedge q$

6) $[p \wedge (p \rightarrow q) \wedge (q \rightarrow r) \rightarrow r]$ is equivalent to
a) T b) F c) R d) ~R
Solution: let $F1 = p$
$F2 = p \rightarrow q$
$F3 = q \rightarrow r$

$F4 = r$

$F1 \wedge F2 \wedge F3 \rightarrow F4$

$\Leftrightarrow \sim (F1 \wedge F2 \wedge F3) \vee F4$ [$X \rightarrow Y = \sim X \vee Y$]

$\Leftrightarrow \sim F1 \vee \sim F2 \vee \sim F3 \vee F4$

$\Leftrightarrow \sim p \vee \sim (p \rightarrow q) \vee \sim (q \rightarrow r) \vee r$

$\Leftrightarrow \sim p \vee \sim (\sim p \vee q) \vee \sim (\sim q \vee r) \vee r$

$\Leftrightarrow \sim p \vee (p \wedge \sim q) \vee (q \wedge \sim r) \vee r$

$\Leftrightarrow (\sim p \vee p) \wedge (\sim p \vee \sim q) \vee (q \wedge \sim r) \vee r$

$\Leftrightarrow T \wedge (\sim p \vee \sim q) \vee (q \wedge \sim r) \vee r$

$\Leftrightarrow (\sim p \vee \sim q) \vee (q \wedge \sim r) \vee r$

$\Leftrightarrow (\sim p \vee \sim q) \vee ((r \vee \sim r) \wedge (r \vee q))$

$\Leftrightarrow (\sim p \vee \sim q) \vee (T \wedge (r \vee q))$

$\Leftrightarrow \sim p \vee \sim q \vee r \vee q$

$\Leftrightarrow \sim p \vee r \vee \sim q \vee q$

$\Leftrightarrow \sim p \vee r \vee T$

$\Leftrightarrow T$

7) The binary operation $\square$ is defined as follows

P	Q	$p \square q$
T	T	T
T	F	T
F	T	F
F	F	T

Which one of the following is equivalent to $p \vee q$?

a) $\sim q \square \sim p$ b) $p \square \sim q$ c) $\sim p \square q$ d) $\sim P \square \sim Q$

Solution: $p \square q \Leftrightarrow q \rightarrow p$

$\Leftrightarrow \sim q \vee p$

$p \vee q \Leftrightarrow q \vee p \Leftrightarrow \sim(\sim q) \vee p \Leftrightarrow p \square \sim q$

Option (b) is correct answer.

8) Consider the following logical inferences

I_1: If it rains then the cricket match will not be played
The cricket match was played.
Inference: There was no rain

I_2: If it rains then the cricket match will not played
It did not rain
Inference: The cricket match was played

Which of the following is TRUE?

a) Both I_1 and I_2 are correct inferences
b) I_1 is correct but I_2 is not a correct inference
c) I_1 is not correct but I_2 is correct inference

d) Both I_1 and I_2 are not correct inferences.

9) F_1: P → ~P F_2: (P → ~P) ∨ (~P → P)

a) F_1 is satisfiable and F_2 is valid

b) F_1 is unsatisfiable and F_2 is satisfiable

c) F_1 is unsatisfiable and F_2 is valid

d) F_1 and F_2 are both satisfiable

Solution: 1. When P = T, ~P = F F1 will become F

It cannot be valid but satisfiable because it can be true when P = F.

2. When P = T then ~P = F but F_2 is true

When P = F then ~P = T but F_2 is true

F_2 is always true that means it is valid.

10) Which of the following is not a valid logical implication?

a) P ∧ (P → Q) ⇒ Q

b) ~P ∧ (P → Q) ⇒ ~Q

c) P ∧ Q ⇒ P ∨ Q

d) (P → Q) ∧ ~Q ⇒~p

Solution: If F_1 ⇒ F_2 and F_1 = true, then F_2 cannot be false.

In option b)

~P ∧ (P → Q) ⇒ ~Q

When Q = T and P = F

~ (F) ∧ (F → T) ⇒ F

T ∧ T ⇒ F

T ⇒ F

Hence option (b) is correct answer

❖ Practice Questions from Propositional Logic:

1) ~ (P ∧ Q) V (~P V Q) ⇔

a) P   b) Q   c) ~P   d) T

2) ~ (P ↔ Q) is equivalent to

a) ~P ↔ ~Q

b) ~P ↔ Q

c) (P ∧ Q) ∨ (~P ∧ ~Q)

d) (P∨ ~Q) ∧ (~p ∨ Q)

3) ((P→Q)∧~Q)→~P is

a) Tautology   b) Contradiction   c) Contingency   d)none

4) Which of the following arguments is invalid?

a) $P \lor Q, \sim P \to R, Q \to S \vdash R \land S$

b) $P \to \sim Q, R \to Q, R \vdash \sim P$

c) $P \to R, Q \to R, Q \lor P \vdash R$

d) $P \to \sim Q, \sim Q \vdash P$

5) Which of the following arguments are invalid?
S1: If it rains Erick will be sick

It did not rain

Erick was not sick

S2: If I study then I will not fail mathematics

If I do not play basket ball, then I will study

But I failed mathematics

Therefore I must have played basket ball

a) Only S1 b) Only S2 c) both S1 and S2 d) neither S1 nor S2

6) $(P \to Q) \leftrightarrow (\sim Q \to \sim P)$ is equivalent to

a) T b) F c) P d) Q

7) $((P \lor Q) \to P) \to (Q \to P)$

a) T b) F c) P d) Q

8) $(P \to Q) \lor (\sim P \to R)$ is equivalent to

a) T b) F c) $Q \lor R$ d) none

9) $(P \leftrightarrow Q) \lor (\sim Q \leftrightarrow R) \lor (\sim R \leftrightarrow P) \Leftrightarrow$

a) T b) F c) $P \land Q$ d) $P \land Q \land R$

10) Which of the following inference system is invalid?

a) $R \to S, \sim S \vdash \sim R$

b) $\sim R, P \to Q, Q \to R \vdash \sim P$

c) $\sim R \to (S \to \sim T), \sim R \lor W, \sim P \to S, \sim W \vdash T \to P$

d) $P \land Q \to \sim T, W \lor R, W \to P, R \to Q \vdash (W \lor R) \to \sim T$

11) If P then Q unless Z is equivalent to

a) $(P \to Q) \lor \sim Z$ b) $P \land Z \to Q$ c) $\sim Z \to (P \to Q)$ d) $(P \to Q) \to \sim Z$

12) Which of the following statement is true?

S1: $(P \rightarrow Q) \vee \sim R \Leftrightarrow P \wedge R \rightarrow Q$

S2: $P \rightarrow (Q \rightarrow \sim Z) \Leftrightarrow \sim(P \wedge Q \wedge Z)$

a) Only S1 is correct
b) Only S2 is correct
c) S1 and S2 both are correct
d) Neither S1 nor S2 is correct

13) Which of the following is not a tautology?

a) $P \wedge Q \rightarrow P$ b) $P \rightarrow P \vee Q$ c) $\sim P \wedge (P \vee Q) \rightarrow Q$ d) $\sim (P \rightarrow Q) \rightarrow Q$

14) Which of the following is a contradiction?

a) $\sim (P \rightarrow Q) \rightarrow Q$

b) $\sim (P \rightarrow Q) \rightarrow P$

c) $\sim (P \rightarrow Q) \rightarrow \sim Q$

d) $\sim (\sim P \vee (\sim P \rightarrow Q))$

15) S1: $P \vee (P \wedge Q) \Leftrightarrow P$

S2: $P \wedge (P \vee Q) \Leftrightarrow Q$

a) Only S1 b) Only S2 c) Both of them are correct d) neither S1 nor S2

16) S1: $P \rightarrow Q \Leftrightarrow \sim Q \rightarrow \sim P$

S2: $\sim (P \leftrightarrow Q) \Leftrightarrow \sim P \leftrightarrow \sim Q$

a) Only S1 b) Only S2 c) both S1 and S2 d) neither S1 nor S2

17) $(P \rightarrow Q) \rightarrow R$ is equivalent to

a) $P \rightarrow (Q \rightarrow R)$ b) $P \rightarrow Q \rightarrow R$ c) $P \rightarrow Q \vee R$ d) none

18) $P \rightarrow (Q \rightarrow R)$ is equivalent to

a) $(P \rightarrow Q) \rightarrow R$ b) $P \wedge Q \rightarrow R$ c) $P \rightarrow Q \wedge R$ d) $P \wedge \sim Q \rightarrow R$

NOTE: The dual of compound proposition that contains only the logical operators $\wedge, \vee, \sim$ is the proposition obtained by replacing each $\vee, \wedge$, by each $\wedge, \vee$. Each T by F and each F by T. The dual proposition of S is denoted by S_d then

19) $(P \vee F) \wedge (Q \vee T)_d \Leftrightarrow$

a) $(P \wedge F) \vee (Q \wedge T)$

b) $(P \wedge T) \vee (Q \wedge F)$

c) $(\sim P \wedge T) \vee (\sim Q \wedge F)$

d) $(\sim P \wedge F) \vee (\sim Q \wedge T)$

20) If S_d is a dual of S then $(S_d)_d \Leftrightarrow$

a) ~S b) S c) T d) F

21) S1: If P, Q, R are three compound propositions and If P does not logically equivalent to Q and Q does not logically equivalent to R then P does not logically equivalent to R

S2: If P, Q, R are three compound propositions and $P \Leftrightarrow Q$ and $Q \Leftrightarrow R$ then $P \Leftrightarrow R$

a) Only S1 b) Only S2 c) both S1 and S2 d) neither S1 nor S2

22) S1: A formula is valid iff its complement is not satisfiable

S2: A formula is satisfiable iff its complement is not valid.

a) Only S1 b) Only S2 c) both S1 and S2 d) none

23) Consider a binary operator □ defined as follows

P	Q	P □ Q
T	T	F
T	F	F
F	T	T
F	F	F

The propositional formula $P \wedge Q$ is equivalent to

a)~P □ ~Q b)~P □ Q c)P □ ~Q d)P □ Q