FIRST ORDER LOGIC

- **Open proposition:** An open proposition (or predicate) in n variables from a set U is a function f: $U^n \rightarrow \{T,F\}$, where U^n denotes the Cartesian product of n copies of the set U and T and F respectively stand for true and false. The set U is called universe of discourse (universe, for short) of the open proposition f.

  Example:

  1. x + y=10
  2. X is a rational number
  3. He is a lawyer and she is a computer scientist

Note: Universe of discourse can be taken appropriately.

- Open propositions are functions of variables, and that when we assign specific values to the variables we obtain "Values" of this function, the latter values of this function being propositions that are either true or false.
- Following notation can be used to indicate open proposition

1. P(x, y): x + y =10
2. R(x): x is a rational number
3. L(x): x is a lawyer
4. C(x): x is a computer scientist.

Open propositions can be combined with logical connectives.

Example: if x is a rational number then y is a prime number

$R(x) \rightarrow P(y)$

Where R(x): x is a rational number

P(y): y is a prime number

- Certain declarative sentences involve words that indicate quantity such as all, some, none or one
  1. Every mathematician is an intelligent
     For all x ∈ mathematician, x is an intelligent
     We can write it as $\forall x\ I(x)$ where x ∈ universe of discourse. Here universe of discourse is mathematicians.
  2. Some birds cannot fly
     There is atleast1 x ∈ set of birds, such that x cannot fly $\exists x, F(x)$.
     Here F(x): x cannot fly.
     Universe of discourse: Birds

$\forall$,$\exists$ are used to quantify the statement. Hence they are called quantifiers.

Example: Translate following statements into First Order Logic

1. Every lion is non-vegetarian

   Domain: Animals

   For all x B ∈ animals if x is lion then x is non-vegetarian.

   $\forall x\, [L(x) \rightarrow NV(x)]$

   Where L(x): x is a lion

   NV(x): x is non-vegetarian

2. Some prime numbers are even numbers

   Where P(x): x is a prime number

   E(x): x is an even number

   There is atleast1 number which is both prime number and even number.
   $\exists x\, [P(x) \wedge E(x)]$

3. Ramu loves Sita

   Where L(x, y): x loves y

   L (Ramu, Sita)

4. Ramu loves every one

   L (Ramu, P1) → Ramu loves person called P1

   L (Ramu, P2) → Ramu loves person called P2

   Ramu loves every one means for every L (Ramu, x)
   X ∈ person
   That means $\forall x$ L (Ramu, x)

5. Everyone loves every one

   $\forall x$ L (Ramu, x) means Ramu loves every one

   $\forall x$ L (Rahim, x) means Rahim loves every one

   Instead of Ramu, Rahim if variable x is substituted then formula will become

   $\forall x\, L(y, x)$ → Person y loves every one.

   We want to write that every y likes everyone.

   So finally it would be $\forall y\, \forall x\, L(x, y)$ which is everyone likes every one.

6. Some one likes every one.

   $\forall x$ L (Ramu, x)

   Ramu likes every one

   Some one likes every one can be achieved by substituting someone in the palce of Ramu

Hence Ramu will be substituted by some variable y.

That means $\exists y\ [\forall x\ L(y, x)]$

7. Someone is liked by every one

   $\exists x\ L(P1, x)$

   Some x is liked by P1.

   But our statement is some x is liked by every one, that means

   $\exists x\ L(P1, x) \wedge \exists x\ L(P2, x)$……….[P1,P2,…..all are replace by $\forall y$]

   $\forall y\ \exists x\ L(y, x)$

8. Nobody loves every one

   Try to write "Ramu does not love everyone". That means

   "It is false that Ramu loves everyone."

   $\neg [\forall x\ L(Ramu, x)]$

   $\forall x\ L(Ramu, x)$ is Ramu loves every one.

   It is not only Ramu, but everyone does the same is our statement.(Remove Ramu and substitute variable y)

   $\forall y\ [\neg \forall x\ L(y, x)]$

   For every y [it is false that y loves every one]

2nd one

   Ramu does not love everyone is equal to Ramu hates atleast1 person.

   That means $\exists x\ \neg L(Ramu, x)$

   Generalize to everyone

   $\forall y\ \ [\exists x\ \neg L(y, x)]$

   Every [that(y) one hates
   One(y) atleast1 person]

❖ Understanding 2 variable first order logic statements with 2 quantifiers:

1. $\forall x\ \forall y\ P(x, y)$
2. $\forall y\ \forall x\ P(x, y)$
3. $\forall x\ \exists y\ P(x, y)$
4. $\exists y\ \forall x\ P(x, y)$
5. $\forall y\ \exists x\ P(x, y)$
6. $\exists x\ \forall y\ P(x, y)$
7. $\exists x\ \exists y\ P(x, y)$
8. $\exists y\ \exists x\ P(x, y)$

What is the logical relationship between the 8 equations?

$$\forall x \, \forall y \, P(x, y) \Leftrightarrow \forall y \, \forall x \, P(x, y)$$

For easy interpretation assume P(x, y) as x and y does some project.

Then $\forall x, \forall y$ P(x, y) becomes every x and every y does the project.

$\forall y \, \forall x$ P(x, y) becomes every y and every x does the project

Where x, y belongs to set of students.

It can be easily seen that above two first order logic statements are logically equivalent.

Consider (3) & (4)

$\forall x \ \exists y$ P(x, y), $\exists y \ \forall x$ P(x, y)

$\forall x \ \exists y$ P(x, y) $\Leftrightarrow \exists y \ \forall x$ P(x, y) is false

To understand it try to understand the meaning or description of above formulas.

- Description of $\forall x \ \exists y$ P(x, y): For every student x, there is atleast one student y so that x&y does project.
- Description of $\exists x \ \forall y$ P(x, y): There is at least one student x, for every student y. so that x & y does project.

Formula 3 claims that every student does project with atleast1 student where formula 4 claims that there is atleast1 student who does project will all the students.

But if 4 is true then 3 is also true. If 3 is true then 4 need not be true.

Example:

	Ram	Rahim	Robert
Ram	√	√	X
Rahim	√	√	X
Robert	X	X	√

Every student has done a project with at least one student but no student has done project will all the students. Hence

(3) $\Rightarrow$ (4) is false

But (4) $\Rightarrow$ (3) is true

Consider the formulas (7) & (8)

(7): $\exists x \, \exists y$ P(x, y)
(8): $\exists y \, \exists x$ P(x, y)

- Description of (7): There is at least one student x, there is at least one student y such that they do project.
- Description of (8): There is at least one student y, there is at least one student x such that they do project.

$\exists x\, \exists y\ P(x, y) \Leftrightarrow \exists y\, \exists x\, P(x, y)$

It can be easily understand that both formulas describe the same thing.

Now we can summarize the relationship between formulas 1 to 8.

$$\begin{array}{ccc}
\forall x\, \forall y\, p(x, y) & \Leftrightarrow & \forall x\, \forall y\, p(x, y) \\
\Downarrow & & \Downarrow \\
\exists x\, \forall y\, p(x,y) & & \exists y\, \forall x\, p(x,y) \\
\Downarrow & & \Downarrow \\
\forall y\, \exists x\, p(x,y) & & \forall y\, \exists x\, p(x,y) \\
\Downarrow & & \Downarrow \\
\exists y \exists x\, p(x,y) & \Leftrightarrow & \exists x \exists y\, p(x,y)
\end{array}$$

- How to write negation for a formula:

1) $\neg\, \forall x\ P(x)$ =?

Try to understand the description of $\forall x\ P(x)$ by assuming P(x) as x has passed physics exam.

$\neg\, \forall x\ P(x) \Leftrightarrow$ it is false that (every x has passed physics)

$\Leftrightarrow$ At least one student failed physics exam.

$\Leftrightarrow \exists x\ \neg\, P(x)$

2) Similarly $\neg \exists x\ P(x) \Leftrightarrow$ It is false that (at least one student has passed physics exam)

$\Leftrightarrow$ It is true that every student has failed physics exam.

$\Leftrightarrow \forall x\ \neg\, P(x)$

$\neg\, \exists x\ P(x) \Leftrightarrow \forall x\ \neg\, P(x)$

3) $\forall x\ (P(x) \vee Q(x))$

Every student has either passed physics exam or chemistry exam.

$\neg\, \forall x\ (P(x) \vee Q(x))$

It is false that (every student has passed physics exam or chemistry exam) that means atleast1 student has failed in both exams.

$\exists x(\neg P(x) \wedge \neg Q(x))$

$\neg \forall x(P(x) \vee Q(x)) \Leftrightarrow \exists x(\neg P(x) \wedge \neg Q(x))$

Similarly $\neg \exists x(P(x) \vee Q(x)) \Leftrightarrow \forall x(\neg P(x) \wedge \neg Q(x))$

Similarly $\neg \exists x(P(x) \wedge Q(x)) \Leftrightarrow \forall x(\neg P(x) \vee \neg Q(x))$

❖ Practice Questions:

1) Consider a domain S= { 1, 2, 3, 4 }
P(x, y): $x * y \geq 2$
Consider the following statements
   I. $\forall x \forall y P(x, y)$
   II. $\forall x \exists y P(x, y)$
   III. $\exists x \forall y P(x, y)$
   IV. $\exists x \exists y P(x, y)$

Solution:

I. $\forall x \forall y P(x, y)$:

$\forall x P(x) = P(e1) \wedge P(e2) \wedge \ldots\ldots P(en)$

Where e1, e2, en are all the elements in the domain

Similarly

$\forall x \forall y P(x, y) = P(1, 1) \wedge P(1, 2) \wedge P(1, 3) \wedge P(1, 4) \wedge P(2, 1) \wedge P(2, 2) \wedge P(2, 3) \wedge P(2, 4) \wedge P(3, 1) \ldots\ldots P(4, 4)$

$P(1, 1) = 1*1 \geq 2 \Rightarrow$ false

$= F \wedge P(1, 2) \wedge P(1, 3) \ldots\ldots\ldots P(4, 4)$

$= F$

II. $\forall x \exists y P(x, y) \Leftrightarrow \exists y P(1, y) \wedge \exists y P(2, y) \wedge \exists y P(3, y) \wedge \exists y P(4, y)$

$\exists y\, p(1, y) \Leftrightarrow P(1, 1) \vee P(1, 2) \vee P(1, 3) \vee P(1, 4)$

Since $P(1, 4) = 1 * 4 \geq 2$, which is true total formula $\exists y P(1, y)$ is true.

$(P(1, 1) \vee P(1, 2) \vee P(1, 3) \vee T \Leftrightarrow T$

Similarly we can prove that $\exists y P(2, y) \Leftrightarrow T$

$\exists y P(3, y) \Leftrightarrow T$

$\exists y P(4, y) \Leftrightarrow T$

Hence $\forall x \exists y P(x, y) \Leftrightarrow T$

III. $\exists x \forall y P(x, y)$; is there any x so that P(x, y) true for all y?

x cannot be 1 since $1*y \geq 2$ cannot be true for all y

x can be 2 since $2x1 \geq 2$

$2x2 \geq 2$

$2x3 \geq 2$

$2x4 \geq 2$

That means P (2, y) is true for all y in the domain hence $\exists x \, \forall y \, P(x, y) \Leftrightarrow T$

IV. $\exists x \, \forall y \, P(x, y)$; can you show atleast one x, atleast one y so that p(x, y) is true.

Yes when x=4 y=4

$x * y \geq 2$ is true.

Hence it is true.

2) Let P(x) be the statement "x spends more than 5 hours every week in class "
Express each of these quantifiers in English.

a) $\exists x \, P(x)$

Solution: Let us assume domain consists all the students

$\Rightarrow$ Atleast one student x, p(x)

$\Rightarrow$ P(x) means x spends more than five hours in the class

$\Rightarrow$ Atleast one student x spends more than five hours in the class.

b) $\forall x \, P(x)$

$\Rightarrow$ Every student x, p(x)

$\Rightarrow$ Every student x, x spends more than five hours in the class

$\Rightarrow$ every student spends more than five hours in the class.

c) $\forall x \sim P(x)$

$\Rightarrow$ ~P(x) = x does not spend more than five hours in the class
$\Rightarrow$ X spends less than five hours in the class.

$\forall x \sim P(x) \Rightarrow$ every student spends less than five hours in the class.

3) Translate these statements into English, where
C(x): x is comedian
F(x): x is funny

Universe of discourse or domain is all people

a) $\forall x \, (C(x) \rightarrow F(x))$
For every person x [if x is comedian then x is funny]
$\Rightarrow$ Every person, if he is comedian then he is funny.

b) $\exists x \, (C(x) \rightarrow F(x))$
$\Leftrightarrow \exists x \, (\sim C(x) \vee F(x))$
Note: $P \rightarrow Q \Leftrightarrow \sim P \vee Q$
$\Rightarrow$ there is atleast one x, x is not comedian or he is a funny.
$\Rightarrow$ Atleast one person is either not comedian or funny.

c) $\exists x \, (C(x) \wedge F(x))$
There is atleast one person who is comedian and funny.

4) Determine the truth value each of these statements?

a) $\forall n \, (n+1 > n)$ universe of discourse : all integers

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Ans:
$\forall n\ (n+1-n > 0)$
$\forall n\ (1 > 0)$
It is true for every n.
Hence truth value is T

b) $\exists n (n = -n)$
Yes, there is one such n i.e. = 0;
Hence statement is true.

5) Translate each of these statements into first order logic statements?

Note: universe of discourse = students in your class

a) Someone in your class can speak Hindi
Solution: $\exists$ x [H(x)]
H(x): x can speak Hindi.

b) Everyone in your class is friendly
Solution: $\forall$x [F(x)]
F(x): x is friendly.

c) No student in your class has taken CAT training?
~ [$\exists$x C(x)]
$\exists$x C(x): There is some student in this class who has taken CAT training.
It is false that there is some student in this class who has taken CAT training
Hence ~ [$\exists$ x C(x)] is correct translation.

6) Write the negation for below statements:

a) Everyone loves every one
Solution: somebody hates somebody

b) Nobody loves everybody
Solution: someone loves every body

c) Somebody loves somebody
Solution: nobody loves somebody

d) Everyone loves some one
Solution: somebody loves nobody.

7) Consider the following statements
S1: $\forall x\ (P(x) \lor A) \equiv (\forall x\ P(x)) \lor A$
S2: $\forall x\ (P(x) \land A) \equiv\ \forall x\ P(X) \land A$
Check whether they are true or false?
Solution:
S1: Assume universe of discourse contains elements {x1, x2, x3}.
$\forall x\ (P(x)) \equiv p(x1) \land p(x2) \land p(x3)$
$\forall x\ [p(x) \lor A] = [P(x1) \lor A] \land [P(x2) \lor A] \land [p(x3) \lor A]$
$\Leftrightarrow$ According to distributive law
$P \lor (Q \land R) \Leftrightarrow (P \lor Q) \land (P \lor R)$
$\Leftrightarrow p(x1) \land P(x2) \land P(x3) \lor A$
$\Leftrightarrow\ \forall x\ P(x)\ V\ A$

S2: $\forall x\ (P(x) \land A) \equiv\ \forall x\ P(X) \land A$

$\forall x\,(P(x) \wedge A) \equiv \;(P(x1) \wedge A) \wedge (p(x2) \wedge A) \wedge (P(x3) \vee A)$
$\equiv\; P(x1) \wedge P(x2) \wedge P(x3) \wedge A \wedge A \wedge A$
$\equiv\; P(x1) \wedge P(x2) \wedge P(x3) \wedge A$
$\equiv\; \forall x\,(P(x) \wedge A)$

Both s1 and s2 are true.

8) Consider the following statements
S1; $\forall x\,[P(x) \vee Q(x)] \Leftrightarrow \forall x\,P(x) \vee \forall x\,Q(x)$
S2: $\exists x\,[P(x) \wedge Q(x)] \Leftrightarrow \exists x\,P(x) \wedge \exists x\,Q(x)$
Solution: Assume that
P(x): x has passed physics exam
Q(x): x has passed chemistry exam.

$\forall x\,[P(x) \vee Q(x)]$:
Description of the above formula is “every student has either passed physics exam or chemistry exam.

$\forall x\,P(x) \vee \forall x\,Q(x)$
Description of the above formula is “every student has passed physics exam or every student has passed chemistry exam.
Whenever LHS is true then RHS need not be true but if RHS is true then definitely LHS is true.
Hence $\forall x\,[P(x) \vee Q(x)] \Longrightarrow \forall x\,P(x) \vee \forall x\,Q(x)$
But $\forall x\,P(x) \vee \forall x\,Q(x) \Longrightarrow \forall x\,[P(x) \vee Q(x)]$ is false

S2: $\exists x\,[P(x) \wedge Q(x)] \Leftrightarrow \exists x\,P(x) \wedge \exists x\,Q(x)$

$\exists x\,[P(x) \wedge Q(x)]$:
There is atleast one student who has passed both physics and chemistry exams.
$\exists x\,P(x) \wedge \exists x\,Q(x)$: some student passed in physics and some students has passed in chemistry.
When every RHS is true then some student has passed physics exam and some student has passed chemistry exam.
Then one student may not exist who passed both exams.
Hence we can conclude that
$\exists x\,P(x) \wedge \exists x\,Q(x) \Rightarrow \exists x\,[P(x) \wedge Q(x)]$ is not true
$\exists x\,[P(x) \wedge Q(x)] \Rightarrow \exists x\,P(x) \wedge \exists x\,Q(x)$ is true.
Hence s2 is false.

9) Negate the following statement?
$\forall x\,[p(x) \rightarrow Q(x)]$
Solution: $\sim\forall x\,[P(x) \rightarrow Q(x)]$
$\sim \forall x\,[\sim P(x) \vee Q(x)]$
$\exists x \sim [\sim P(x) \vee Q(x)]$ $(\sim \forall x\,F(x) \Leftrightarrow \exists x \sim F(x))$
$\exists x\,[P(x) \wedge \sim Q(x)]$ (demorgan’s law)

10) Convert the following statements into first order logic statements?
“If a person is female and is a parent then this person is some one’s mother “
Solution: This statement is referring every person.
$\forall x$ [x is person, x is female then definitely there is some y, so that x is mother of y]
$\forall x\,[P(x) \wedge F(x) \rightarrow \exists y\,(M(x, y))]$

Where p(x): x is a parent

F(x): x is a female

M(x, y): x is mother of y.

11) $\exists x\ \forall y\ \forall z\ ((F(x, y) \wedge F(x, z) \wedge (y \neq z)) \rightarrow \sim F(y, z))$ translate the statement?

Solution: There is some x such that for every y and z (some formula is true)

$\Rightarrow$ If $F(x, y) \wedge F(x, z) \wedge F(y \neq z) \Rightarrow \sim F(y, z)$

$\Rightarrow$ there is some x, for every y and z

If x is friend of y and x is friend of z and y and z are different people then y, z are not friends.

$\Rightarrow$ there is someone who has friends who are not friends to each other

❖ Practice Questions:

1) Convert the following statements into First order logic?
Some children will eat any food
C(x): means x is a child
F(x): "x is food "
Eat (x, y): x eats y
   a) $\exists x\ \ \forall y\ \ (C(x) \wedge F(x, y) \wedge F(y))$
   b) $\forall x\ \ \exists y\ \ (C(x) \wedge F(x, y) \wedge F(y))$
   c) $\exists x\ [C(x) \wedge\ \ \forall y(F(y) \rightarrow Eat(x, y))]$
   d) $\exists x\ [C(x) \rightarrow\ \ \forall y (F(y)\ \wedge Eat(x, y)]$
2) $\forall x\ [p(x) \rightarrow Q(x)]\ \ \wedge \forall x\ [p(x)\rightarrow R(x)]$ is equivalent to
   a) $\forall x\ [p(x) \rightarrow Q(x) \wedge R(x)]$
   b) $\forall x\ [p(x) \rightarrow Q(x) \vee R(x)]$
   c) $\forall x\ [(p(x) \rightarrow Q(x)) \rightarrow R(x)]$
   d) $\exists x\ [p(x) \rightarrow Q(x) \vee R(x)]$
3) " Tortoises defeats rabbits "
Convert it into first order logic?
   a) $\forall x\ \forall y\ [tortoise (x) \wedge rabbit (y) \wedge defeat (x, y)]$
   b) $\forall x\ \exists x\ [tortoise(x) \wedge rabbit (y) \wedge defeat (x, y)]$
   c) $\forall x\ \forall y\ [tortoise(x) \wedge rabbit (y) \rightarrow defeat (x, y)]$
   d) $\exists x\ \exists y\ [tortoise (x) \wedge rabbit (y) \rightarrow defeat (x, y)]$
4) Convert the following statement into first order logic. "Children do not go to school whenever they are unwell."
Here child(x) means x is a child.
Unwell(x, y) means x is unwell on day y
Location (x, y, z) means location of x on day y is z.
   a) $\forall x\ [child(x) \wedge unwell(x, y)\rightarrow location(x, y, home)]$
   b) $\exists x\ \ [child(x) \wedge unwell(x, y)\rightarrow location(x, y, home)]$
   c) $\forall x\ [child(x) \wedge location(x, y, home)\rightarrow unwell(x, y)]$
   d) $\exists x\ [child(x) \wedge location(x, y, home)\rightarrow\sim unwell(x, y)]$
5) Convert the following statement into first order logic statement?

"All towers are of the same color"
Tower (x): x is a tower.
Color(x): x is of color y.
a) $\forall x \exists y$ [Tower (x) → color (x, y)]
b) $\exists x \forall x$ [color (x, y) → tower (x)]
c) $\exists y \forall x$ [Tower (x) → color(x, y)]
d) $\forall x \exists y$ [color (x, y) → tower (x)]

6) S1: $\forall x \exists y \forall z$ [x+ y = z]
S2: $\exists x \forall y \exists z$ [x + y = z]
Where x, y, z are real numbers. Which of the following statement is true?
a) Only s1
b) Only s2
c) Both s1 and s2
d) None

7) $\forall x P(x) \wedge \exists x Q(x)$ is equivalent to
a) $\forall x \exists y$ [P(x) ∧ Q(y)]
b) $\exists x \forall y$[P(x) ∧ Q(y)]
c) $\exists x \exists y$[P(x) ∧ Q(x)]
d) $\forall x$[P(x) →Q(x)]

8) Which of the following statements are correct?
S1: $\forall x [p(x) \vee Q(x)] \Leftrightarrow \forall x\, p(x) \vee \forall x\, Q(x)$
S2: $\forall x [p(x) \wedge Q(x)] \Leftrightarrow \forall x\, p(x) \wedge \forall x\, Q(x)$
a) Only s1
b) Only s2
c) Both s1 and s2
d) None

9) Consider the following statements
S1: $\exists x [p(x) \vee Q(x)] \Leftrightarrow \exists x\, p(x) \vee \exists x\, Q(x)$
S2: $\exists x [p(x) \wedge Q(x)] \Leftrightarrow \exists x\, p(x) \wedge \exists x\, Q(x)$
Which of the following statements are true?
a) Only s1
b) Only s2
c) Both s1 and s2
d) None

10) Consider the following statements
S1: $\exists x [p(x) \rightarrow Q(x)] \Leftrightarrow \exists x\, p(x) \rightarrow \exists x\, Q(x)$
S2: $\exists x\, p(x) \rightarrow \exists x\, Q(x) \Leftrightarrow \exists x [p(x) \rightarrow Q(x)]$
Which of the following statements are correct?
a) Only s1
b) Only s2
c) Both s1 and s2
d) None

11) Negate the following first order logic statement?
$\forall x \exists y [P(x) \rightarrow Q(y)]$
a) $\exists x \forall y$[P(x) → Q(y)]

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b) $\exists x \forall y [P(x) \wedge \sim Q(y)]$
c) $\exists x \forall y [Q(y) \rightarrow p(x)]$
d) $\exists x \forall y [p(x) \wedge Q(y)]$

12) Consider the following description
P(x): x has passed physics exam
Q(x): x has passed chemistry exam
"There is exactly one person who has passed both exams"
a) $\forall x \forall y [P(x) \wedge Q(y) \wedge x = y]$
b) $\forall x \forall y [P(x) \wedge Q(y) \rightarrow x = y]$
c) $\forall x \forall y [x= y \rightarrow p(x) \wedge Q(y)]$
d) $\forall x \forall y [x\neq y \rightarrow \sim p(x) \wedge \sim Q(y)]$

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