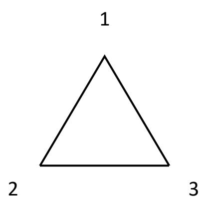
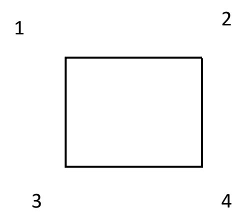
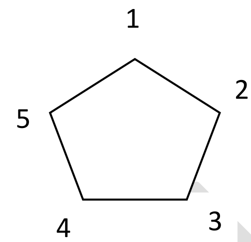
1. Cyclic graph:
A graph on n vertices is called cyclic graph iff it is a cycle on n-vertices.

Example:

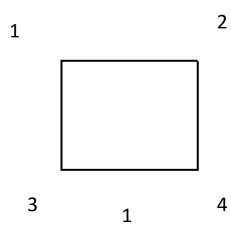




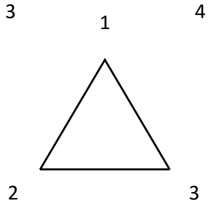


2. K-regular graph:
If every vertex has same degree k then the graph is called k- regular graph.

Examples of a 2- regular graph:

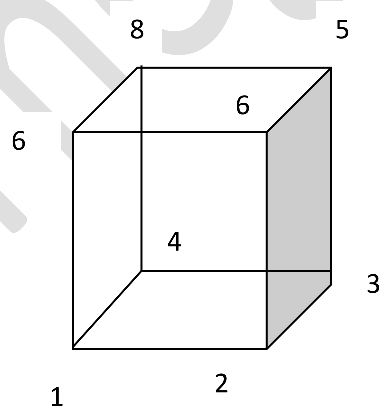


d (1) =d (2) =d (3) =d (4) =2



d (1)=d (2)=d (3)=2

Examples of a 3 regular graph



d(1)=d(2)=d(3)=d(4)=d(5)

=d (6) =d (7) =d (8) =3

3. Complete graph (K_n):
An undirected graph on n vertices is called complete graph if there is an edge between every 2 vertices.

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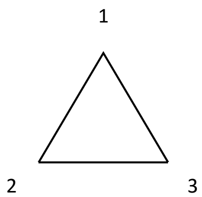
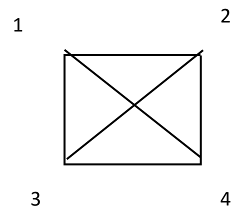
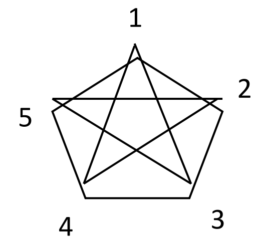
Example:

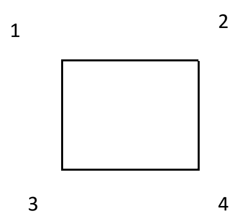


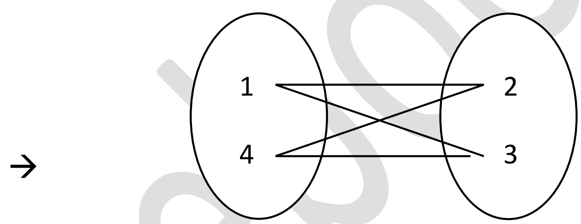


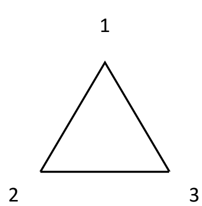


4. Bipartite graph ($B_{m, n}$): A graph G(V,E) is called bipartite graph $B_{m, n}$ where vertex set v can be partitioned into 2 partitions p1,p2 so that |p1|=m, |p2|=n and there is no edge between vertices of same partition.

Example:







It is not bipartite because it cannot be partitioned as explained

5. Complete bipartite graph: ($K_{m, n}$): Bipartite graph $B_{m, n}$ is called complete bipartite graph if every 2 vertices of different partitions has an edge

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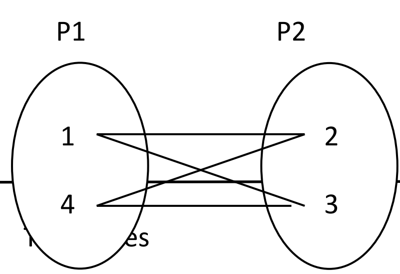
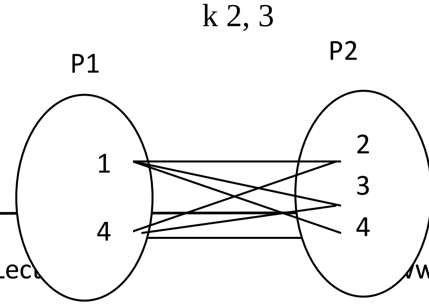
Example

K 2, 2

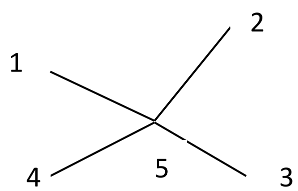


k 2, 3

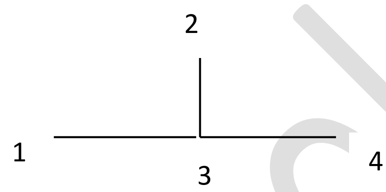


6. Star graph: G (V, E) is called star graph iff every other vertex is connected to a center vertex.

Example:



S5



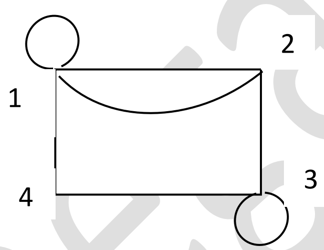
S6

- **Sum of the degrees theorem**: In a graph G (V, E): Sum of all the degrees of vertices is twice the number of edges.

  $\sum d(V_i) = 2*|E|$, where $V_i \in V$

  It is applicable for every undirected graph (multi graphs, graphs with loops)

  Example:



d (1) =2+1+1+1=5

d (2) =1+1+1=3

d (3) =2+1+1=4

d (4) =1+1=2

$\sum d(V_i) = d(1) + d(2) + d(3) + d(4) = 5+3+4+2=14$

Number of edges = 7

$\sum d(V_i) = 2*|E| = 2*7=14$

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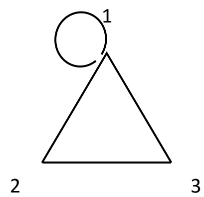
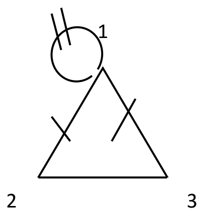
- **Theorem:** sum of degrees of all vertices is equal to twice the number of edges present.

  Proof: since every edge is counted twice in degree calculation

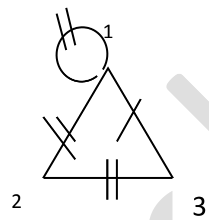
  Each edge is incident on exactly two vertices. Loop is taken as 2 in degree calculation. Hence effectively each edge is counted twice.

  Example:

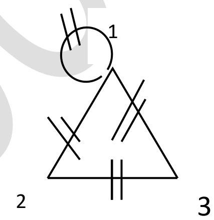




Count degree of 1



Count degree of 2



Count degree of 3

Hence finally each edge is marked twice

- **Theorem:** In a graph G, the number of odd degree vertices is even

  Proof by contradiction:

  Assume that the number of odd-degree vertices is odd.

  Divide the vertices into 2 parts p1, p2

  P1: set of vertices of odd- degree

  P2: set of vertices of even-degree

  Now calculate sum of odd degree vertices. It is $\sum_{\substack{\text{odd degree}\\ \text{vertices}}} d(vi)$

  Now calculate sum of even- degree vertices that is $\sum_{\substack{\text{even degree}\\ \text{vertices}}} d(vi)$

$$\sum_{\substack{\text{odd degree}\\ \text{vertices}}} d(vi) + \sum_{\substack{\text{even degree}\\ \text{vertices}}} d(vi) = \sum_{\text{every vertex}} d(vi) = 2*|E| = \text{even}$$

$$\sum_{\substack{odd\ degree\\ vertices}} d(vi) + \sum_{\substack{even\ degree\\ vertices}} d(vi) = \text{even}$$

Odd + even = even

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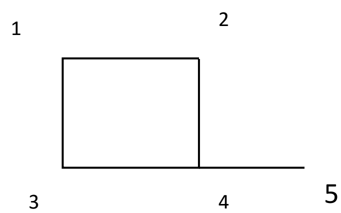
Odd = even

Contradiction, which means our assumption that the number of odd degree vertices is odd is wrong. Hence the number of odd- degree vertices is even.

- **Degree sequence** : $<d_1, d_2, d_3 \ldots d_n>$

If $v_1, v_2, v_3 \ldots v_n$ are the vertices of G, then the sequence $(d_1, d_2, d_3 \ldots d_n)$ is the degree sequence. Usually reorder the vertices such that $d_1 \le d_2 \le d_3 \ldots \le d_n$.

Example:



Degree sequence : <1, 2, 2, 2, 3>

d (1) = 2 d (2) =2

d (3) =2 d (4) =3 d (5) =1

- **Havel- Hakim's theorem:** There is a simple graph with degree sequence $<d_1, d_2, d_3 \ldots d_n>$ where $d_1 \ge d_2 \ge d_3 \ge \ldots d_n$ iff There is a simple graph with $<d_1^1, d_2^1, d_3^1 \ldots d_{n-1}^1>$

Where $d_i^1 = d_i^i = 1, 2, 3 \ldots n-d_{n-1}$

$d_i^1 = d_i - 1$ where $i = n-dn \ldots n-1$

Example: Is there any simple graph with degree sequence < 1, 1, 1, 2, 2 >

There is a simple graph with <1, 1, 1, 2, 2> iff there is a simple graph with $< d_1^1, d_2^1, d_3^1, d_4^1 >$

Where $d_i^1 = d_i$ where $i = 1, 2, 3, 4 \ldots n-d_{5-1}$

$d_i^1 = d_i$ where i=1, 2

$d_1^1 = d^1$; $d_2^1 = d_2$; $d_3^1 = d_{3-1}$; $d_4^1 = d_{4-1}$

$d1^1 = 1$, $d2^1 = 1$, $d3^1 = 0$, $d4^1 = 1$

Now is there any simple graph with <0, 1, 1, 1>?

There is a simple graph with <0, 1, 1, 1> where $0 \ge 1 \ge 1 \ge 1$ iff there is a simple graph with $<d_1^1, d_2^1, d_3^1>$

Note: observe the previous process. We removed 1 from exactly d_n vertices, which are immediately left to d_n

We have <0, 1, 1, 1 >

Now to get new degree sequence remove 1 from exactly d_n (1) vertices, which are immediately left to d_n

New degree sequence is <0, 1, 1-1>

<0, 1, 0>

Arrange it into ascending order (theorem works for only degree sequences with ascending order)

Is there any simple graph with degree sequence <0, 0, 1>

If we remove 1 from above degree sequence then we get <0, -1>

No simple graph with degree sequence <0, -1>. Hence no graph with <0, 0, 1>

Therefore there is no graph with < 0, 1, 1, 1>

Finally no graph exist with the given degree sequence <1, 1, 1, 2, 2>

❖ practice problems

1. The total number of edges present in complete graph k_n are
   a) n(n-1)   b) n2   c) n(n-1)/2   d) n(n+1)/2

   Solution: For every 2 different vertices there is one edge.

   Number of 2-size different combinations = no. of edges

   Number of 2-size different combinations= ${}^{n}C_{2}$ where n = no. of vertices

   ${}^{n}C_{2}$=n! / ((n-2)! 2!) =n (n-1)/2

2. The number of edges present in complete bipartite graph $k_{m, n}$ are

   Solution: each edge is in the form of (a, b) where a∈A, b ∈ B

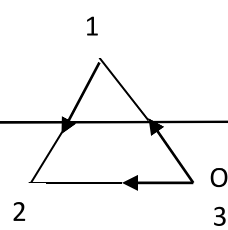
   The number of elements (a, b) denotes the no. of edges.

   The number of elements form Cartesian product between A, B. i.e. AXB

   The number of elements in AXB =|A||B|=m*n hence $|k_{m, n}|$=m*n

3. prove that in a directed graph $\sum d^{+}(v_i)=\sum d^{-}(v_I)=|E|$

   Solution:



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In a directed graph every edge ($v_i \longrightarrow v_j$) is incident towards exactly one vertex and incident from exactly one vertex.

Hence edge is calculated exactly once in in- degree calculation of some vertex and exactly once in out-degree calculation of some vertex.

Hence $\sum d^+(v)$ = sum of in degree vertices = |E|

Similarly $\sum d^-(v)$ = sum of out degree vertices = |E|

4. The number of different graphs possible on n labeled vertices is
   a) $2^{n(n-1)/2}$ b) 2^{n-1} c) n(n-1)/2 d) 2^n

Solution: let us consider a graph with 4 vertices.

The number of edges possible are {1,2}, {1, 3}, {1, 4}, {2, 3} , {2, 4}, {3, 4}

Let k different graphs are possible with these edges

It can be represented as follows, where '0' indicates no edge and '1' indicates presence of edge

	{1, 2}	{1, 3}	{1, 4}	{2, 3}	{2, 4}	{3,4}
G1	0	0	0	0	0	1
G2	1	0	0	0	0	0
G3	1	1	1	0	0	0
	.					
Gk	1	1	1	1	1	

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We can see each graph is formed by writing a binary sequence.

The number of different binary sequences = number of different graphs = 2^x

Where x = length of the binary sequence

Length= All possible edges in the complete graph = $^nC_2$ = n (n-1)/2

Hence the number of different graphs = $2^{n(n-1)/2}$

5. The number of edges in k- regular graph with n-vertices?
   a) n * k b) (n*k) / 2 c) n+ k d) n^2-k^2

Solution: we know degree of every vertex and the number of vertices.

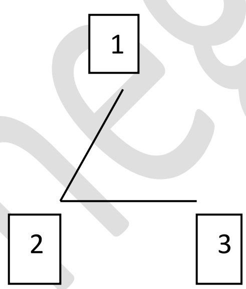
By sum of the degrees theorem ∑ d (vi) = 2*|E|

k+ k+ k+…..n times = 2|E|

|E|=(n*k) /2

6. The number of different graphs on n vertices and k edges is -------?
   Solution: Graph on n vertices can be viewed as binary sequence of n (n-1)/2 positions

Example:



{1, 2}	{2, 3}	{1, 3}
1	1	0

Here 1 represents presence of an edge and 0 represents absence of edge.

A graph with n vertices and k edges can be viewed as binary sequence of

n (n-1)/2 length

The no. of binary sequences of length n (n-1)/2 with exactly k is $^{n(n-1)/2}C_k$

7. What is the largest possible number of vertices in a graph with 35 edges and all vertices of degree atleast3?

Solution:

Let there are n vertices v_1, v_2, v_3….v_n

Degree of v_1=3 + k_1 (k_1≥0)

Degree of v_2 = 3 + K_2 (k_2≥0)

. Degree of v_n = 3 + k_n (k_n ≥ 0)

By sum of degrees theorem

$\sum d(v_i) = 2^* |E|$

Given degree is atleast3 so

$\sum d(v_i) \leq 70$

3n≤70

n≤23.33

n=23

8. Let δ(G) and Δ(G) denote the minimum and maximum degrees of all the vertices of G. Show that for a non-directed graph δ(G)≤ 2*|E|/|V| ≥Δ(G)

Solution: apply sum of degrees theorem

$d(v_1) + d(v_2) + \ldots d(v_n) = 2|E|$

Since δ(G) ≤ $d(v_i)$ for i=1,2….n

$\delta(G) + \delta(G) + \ldots \delta(G) \leq 2^*|E|$

$|V| * \delta(G) \leq 2^*|E$ [here |V| = no. of vertices]

$\delta(G) \leq 2^*|E|/V$

Similarly $\Delta(G) \geq d(v_i)$ where i=1, 2, 3…n

$d(v_1) + d(v_2) + \ldots d(v_n) = 2^* |E|$

$\Delta(G) + \Delta(G) + \ldots \Delta(G) \geq 2^*|E|$

$|V|^* \Delta(G) \geq 2^*|E|$

$\Delta(G) \geq 2^*|E|/|V|$

Hence $\Delta(G) \geq 2^*|E|/|V| \geq \delta(G)$

9. Let G be a non – directed graph with 9 vertices such that each vertex has degree 5 or 6. Prove that atleast5 vertices have degree 6 or atleast6 vertices have degree 5

Proof:

deg 6	deg 5
0	9*
1	8
2	7*
3	6
4	5*

In each case except the last case there are 6 vertices of degree 5. However the cases marked with * are not possible. (no of odd degree vertices must be even in any graph). So the last case is not possible. (That is the only case where assertion given in the problem becomes false.)

So we can say that atleast6 vertices have degree 5 and atleast5 vertices of degree 6

10. Is there any simple graph with degree sequence <1, 1, 1, 1, 2, 2, 3, 3, 3, 3>

Solution: <1, 1, 1, 1, 2, 2, 3, 3, 3, 3>

<1, 1, 1, 1, 2, 2, 2, 2, 2> [by removing vertex with degree 3]

<1, 1, 1, 1, 1, 1, 2, 2> [by removing vertex with degree 2]

< 0, 1, 1, 1, 1, 1, 1,> [by removing vertex with degree 2]

<0, 0, 1, 1, 1, 1> [by removing vertex with degree 1]

< 0, 0, 0, 1, 1> [by removing vertex with degree 1]

< 0, 0, 0, 0> [by removing vertex with degree 1]

Since <0, 0, 0, 0> is graphic every sequence in this degree sequence is also graphic

So <1, 1, 1, 1, 2, 2, 3, 3, 3, 3> is possible.

PRACTICE QUESTIONS -2

1. Suppose that G is a non-directed graph with 12 edges. Suppose that G has 6 vertices of degree 3 and the rest have degrees less than 3. The minimum number of vertices G can have?
   a) 2 b) 0 c)1 d)3
2. A sequence d = <d_1, d_2, d_3… d_n> is graphic if there is a simple non-directed graph with degree sequence d then which one of the following sequences is graphic?
   a) (2, 3, 3, 4, 4, 5)
   b) (1, 3, 3, 3)
   c) (2, 3, 3, 4, 5, 6, 7)
   d) (2, 3, 3, 3, 3)
3. The number of edges in star graph S_n?
   a) n+1 b) n c) n-1 d) n-2
4. A graph G has k isolated vertices and n + k vertices. The maximum number of edges graph G can have?
   a) n(n-1) b)n(n-1)/2) c) n(n-k+1)/2 d) n(n+k-1)/2
5. Let G be a graph and M_G is an adjacency matrix representation of graph G containing n vertices. In adjacency matrix every row has odd number of 1's. Which one of the following cannot be the number of vertices in a graph?
   a) 4.

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b) 6
c) 7
d) 8

6. Let M_G represents adjacency matrix representation of the simple undirected graph with n edges.

Which one of the following statement cannot be true?

a) Diagonal entries are all 0's
b) Sum of the entries of a matrix = 2n
c) It is a symmetric matrix
d) None of the above

7. Which one of the following graph cannot be bipartite graph ?

a) S_5 b) C_4 c) K_4 d) E_4

8. The number of vertices in a 4-regular graph with 32 edges?

a) 8 b) 16 c) 24 d) 4

9. G is a bipartite graph with 2n vertices then maximum number of edges G can have?

a) 2n b) n (2n-1) c) $4n^2$ d) n^2

10. Consider the statements

S1: every cycle graph is a 2-regular graph

S2: every 2 regular graph is cycle graph

a) Only s1
b) Only s2
c) Both s1and s2
d) Neither s1 nor s2

11. A graph has a degree sequence <1, 1, 2, 2, 3, 3, 3, 3> the number of edges in the graph?

a) 18 b)9 c)36 d)8

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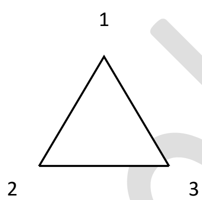
CONNECTIVITY

- **Path**: In a graph G (V, E) sequence of edges (v0, v1) (v1, v2) (v2, v3) … is called path

  Example:

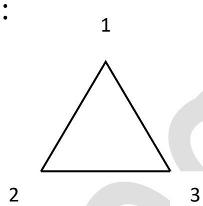
  (1, 2)(2, 3)
  (1, 2)(2, 3)(3, 2)(2, 1)
  and so on
  (1, 2)(2, 3) can be written as 1-2-3 and
  (1, 2) (2, 3) (3, 2)(2, 1) can be written as 1-2-3-2-1

- **Simple path**: Path without the repetition of edges

  Example: 1-2-3-1 and 1-2-3 are simple paths but not 1-2-3-2 where an edge (2, 3) is repeated.

- **Connected graph**: A graph is called connected graph iff there is a path between every two vertices

  Example:

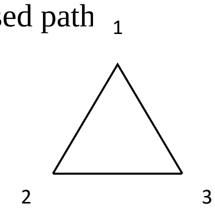
  

  between 1 to 2 there is a path 1-2 or 1-3-2
  Between 1 to 3 there is a path 1-2-3 or 1-3
  Between 2 to 3 there is a path 2-3 or 2-1-3

  Hence this graph is connected

- **Closed path**: A path where start vertex = end vertex

  Example: 1-2-3-1 is called closed path

- **Open path**: A path where start vertex ≠ end vertex

  Example: 1-2-3

- **Circuit**: A simple closed path is called circuit

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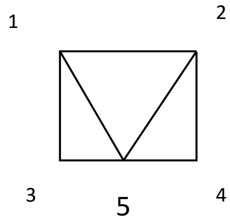
- **Cycle**: A simple closed path without repetition of vertices (possibly at end).

Examples:



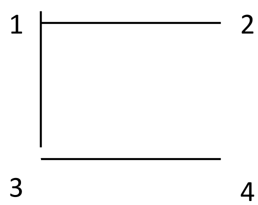
1-3-5-4-2-1 is a circuit and also cycle

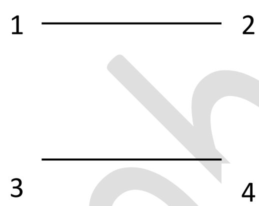
1-3-5-4-2-5-1 is a circuit but not a cycle since

Vertex 5 is repeated

- **Spanning sub graph of** G (V, E): $G^1 (V^1, E^1)$ is called spanning sub graph of G if $V^1 = V$ and $E^1 \subseteq E$

Example:





V= {1, 2, 3, 4}

E= {{1, 2}, {1, 3}, {3, 4}}

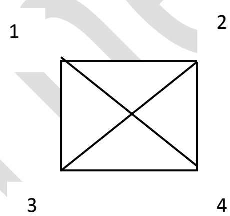
v1 = {{1, 2}, {3, 4}}

E1= {{1, 2}, {3, 4}}

$V^1 = V$ and $E^1 \subseteq E$

- **Induced sub graph of G (V, E)**: $G^1 (V^1, E^1)$ is called induced sub graph of G if $v^1 \subseteq V$ and E^1 is all the edges of E which are defined between vertices of V^1.

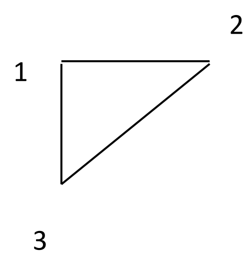
Example:



G (V, E)

V= {1, 2, 3, 4}

E={{1,2},{1,3},{1,4},{2,3},{2,4},{3,4}}



$G^1 (V^1, E^1)$

V1= {1, 2, 3}

E1={{1,2},{1,3},{2,3}}

$V^1 \subseteq V$ and E1 contains all the edges of G which are defined on { 1,2,3 }.

- **Sub graph of G (V, E)**: G1 (V1, E1) is called sub graph of G if v1 ⊆ v and E1 ⊆ E

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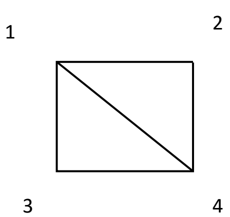
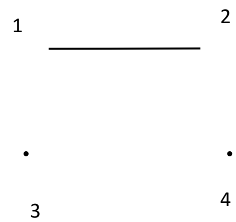


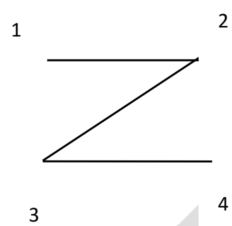


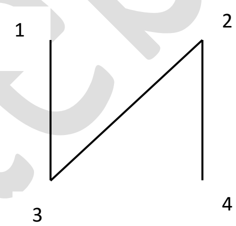
G (V, E) G1(V1, E)

Here G1 ⊆ G

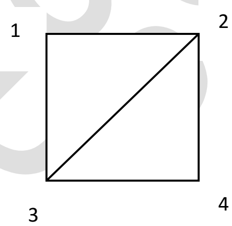
- ❖ Operations on graphs:
  - ➢ **Union**: let G (V, E1) and G2 (V, E2) are graphs then G1UG2 is also a graph. It is defined as G1UG2 (V, E1UE2)

Example:





G1UG2 =

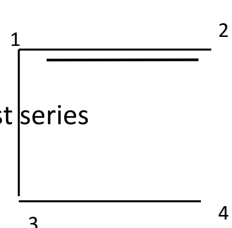


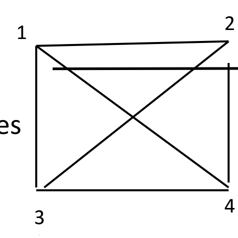
That means all the edges of E1 and all the edges of E2 become edge set of G1UG2

- ➢ **Intersection**: G1∩ G2 (V, E1 ∩E2)

Example:

E1 E2





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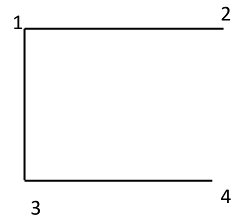
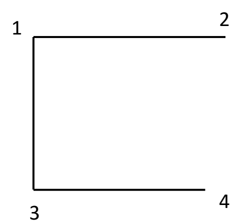
E1 E2



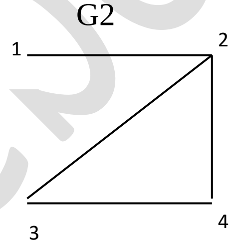
➢ Minus: G1-G2 = (V, E1-E2)

Example:

G1



G2



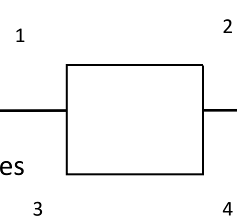
G1 – G2

❖ Connected components: In a graph G, C ⊆ G is called connected component of G if

(i) There is no C1 ⊃ C such that c1 is connected.

➢ Removal of vertex from G: G-{v}

Vertex removal is removing vertex and all the associated edges from the graph.



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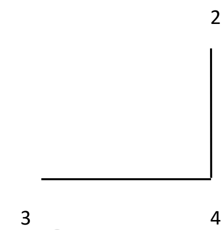
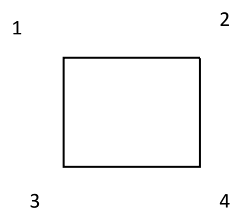
G -{1} =

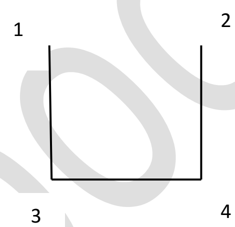


- **Removal of an edge e**: Edge removal is removing an edge from the graph.



G

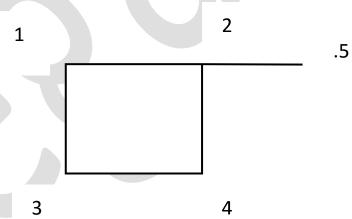
- {1, 2} =

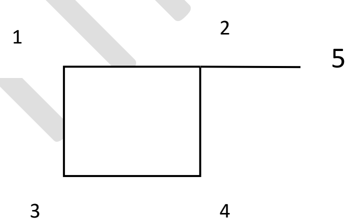


G – {1, 2}

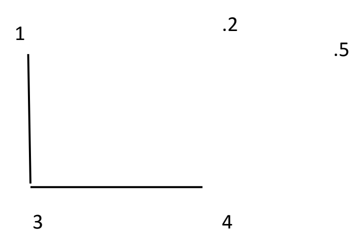
- **Cut vertex:** Cut vertex is a vertex whose removal either disconnects the connected graph or increases the number of connected components.

  Example: 2 is called cut vertex.





-{2}=



Connected graph

Disconnected graph

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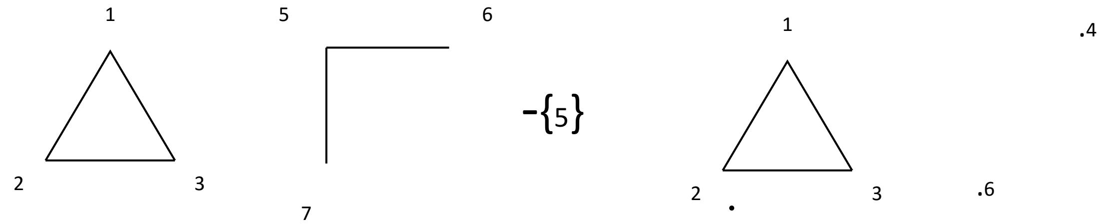
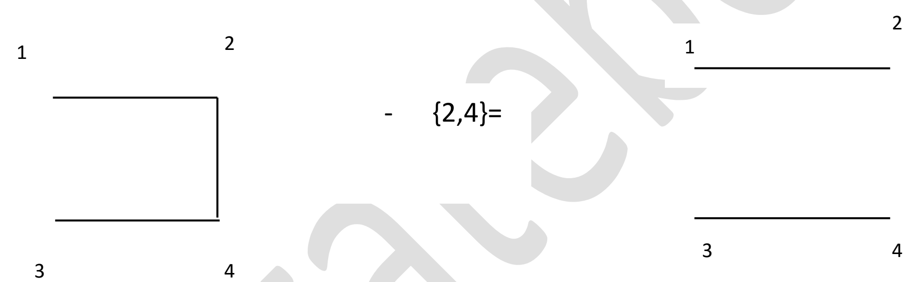


Disconnected graph

the number of components are increased

Hence 5 is also cut – vertex.

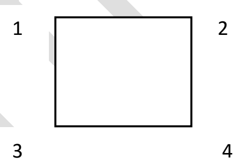
- **Cut edge**: Cut edge is an edge in the graph, whose removal either disconnects the connected graph or increases the number of connected components.



Removal of {2, 4} disconnects the graph.

- **Cut vertex set**: cut –vertex set is a set of vertices whose removal either disconnects the graph or increases the number of connected components.

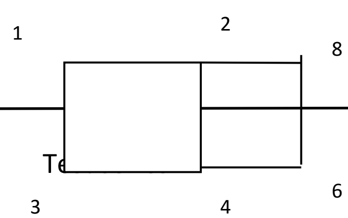
Example



Removal of {1, 3} disconnects the graph

Similarly {2, 4} disconnects the graph.

Example:



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{1, 4} disconnects the graph

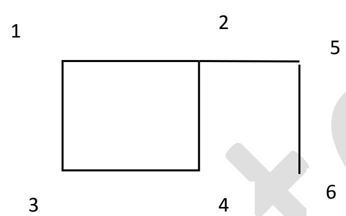
{1, 4, 6} disconnects the graph

{3, 2, 5} disconnects the graph

{3, 2, 5, 6} disconnects the graph

- Cut edge Set: cut edge set is a set of edges whose removal either disconnects the graph or increases the number of connected components.

  Example



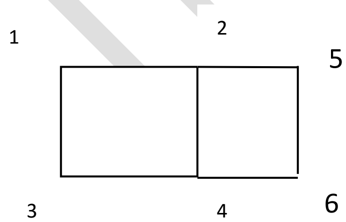
Edge {{2, 5}} is cut – edge set

Similarly {{1, 3}, {2, 4}} is cut – edge set.

{{1, 3},{ 2, 4 }, { 5, 6}} is also cut – edge set

- Vertex connectivity number (K): Minimum value among sizes of all the cut-vertex sets is called vertex connectivity number

  Example:



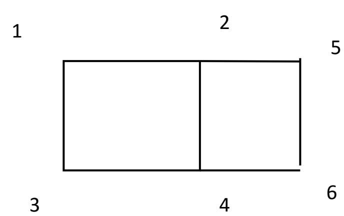
At least 2 vertices are required to disconnect above graph. Hence 2 is minimum among all the sizes of cut-vertex sets.

- **Edge connectivity number (λ)**: Minimum value among sizes of all the cut- edge sets is called edge connectivity number.

  Example:

  At least 2 edges are required to disconnect the graph. Hence 2 is the minimum value among all the sizes of cut edge sets.

  Hence edge connectivity number is 2.



❖ Practice problems with explanations

1. To have a graph with n vertices and k connected components, the number of edges required is at least n-k.
   Solution: Imagine a graph with n vertices and k connected components.

   . . . .
   1 2 3 4 5 6 n

   Fig: E_n

   To have k connected components in a n vertex graph, we need at least 0 edges.
   To have 1 less than n connected components at least 1 edge is required.(we can merge 2 components with one edge)

   . . . .
   1 2 3 4 5 6 n

   To have n-2 connected components at least 2 edges are required.
   To have n-(n-k) = k connected components at least n-k edges are required.
2. The number of edges in a graph with 5 vertices is 7 then the number of vertices in complement graph is?
   Solution:
   G^I = K-G
   The edges of G^I are the edges of complete graph but which are not in G.

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The total number of vertices = 5.

The number of edges in complete graph = $5C_2$ = 10.

Edges of G + edges of G^I = edges of K5

7+x=10

X=3

Number of edges in G^I = 3

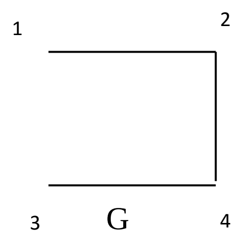
3. Consider the following 2 statements

   S1: complement of every connected graph is disconnected.

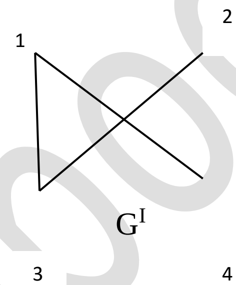
   S2: complement of every disconnected graph is connected.

Solution:

s1



G



G^I

Counter example for s1

S2:-

Let us take G as disconnected, then it has k connected components where (k>1).

Case: 1

Now if u,v are vertices which belongs to different components in G then u and v are connected in G. Since there is no edge between u and v then there will be an edge between u,v in G

Case 2:

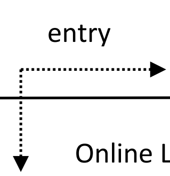
If u, v belongs to same partition then consider a vertex w which is from different partition in G Since there is no edge between u and w in G so there is an edge (u-w) between them in G^I since there is no edge between v and w in G so there is an edge (v-w) between them in $G^{I|}$.

u-w and w-v make a u-v path. So there is a path between them G^I. Hence G^I is connected.

Only s2 is correct choice

4. Show that in any graph there is a non trivial path from any vertex of odd degree to some other vertex of odd degree?

Solution : structure of even degree vertex



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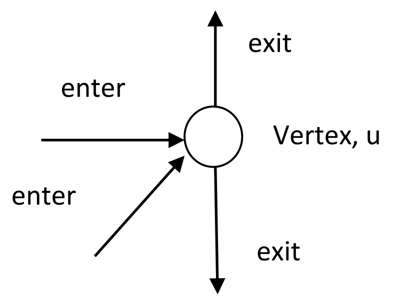
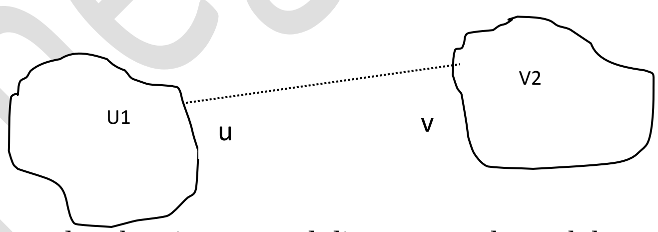
exit

Since the number of edges associated to even degree vertex is even whenever I visit some vertex through some edge definitely there would be an edge to leave the vertex



Now consider a graph G. Let's assume it has an odd degree vertex u. 'u' can have at least one edge. When you take that edge you would go to new vertex v_1. If v_1 is odd degree vertex then theorem is correct, otherwise it is even and just we entered into v_1 so there is an edge to leave. Now we would go to vertex v_2. If that vertex is odd- degree vertex then there is nothing to prove. If it is even-degree vertex then again we can visit new vertex with an exit edge which is corresponding to recent entering edge. (We are ensuring that every time we are visiting new edge). Since graph can only have finite number of edges, after some time we have to stop it(why?). when you finally stop somewhere in the graph, If it is even degree vertex then you were already entered using an edge. So definitely there would be leaving edge. So we can say that only way to stop is at odd degree vertex.

5. Show that an edge in a simple graph is a cut-edge if and only if this edge is not part of any simple circuit in the graph?
   Solution:



If u-v is a cut-edge then its removal disconnects the path between at least 2 vertices .
If x and y are such vertices we have to show

1. If u-v is in circuit $\Rightarrow$ u-v is not cut-edge
2. If u-v is cut-edge $\Rightarrow$ u-v is not in circuit

Case 1: If u-v is in circuit then between u, v there are at least two paths. Though u-v is removed we can reach u to v from alternate path

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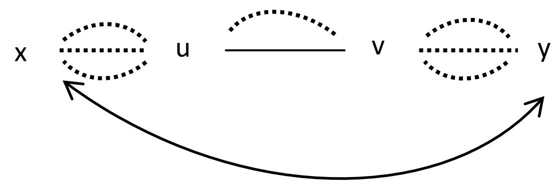
Case 2: If u-v is cut-edge then atleast 2 vertices would have no path if u-v is removed

x and y are such vertices



If u-v is in any circuit then removal of u-v may not separate x and y hence u-v should not be in a circuit..

6. Show that if a simple graph G has K connected components and these components have n1, n2, n3 vertices respectively then the number of edges of G does not exceed-------------

Solution:

If each connected component is full, that means complete graph then the number of edges in the graph = ${}^{n1}C_2 + {}^{n2}C_2 + \ldots\ldots + {}^{nk}C_2$

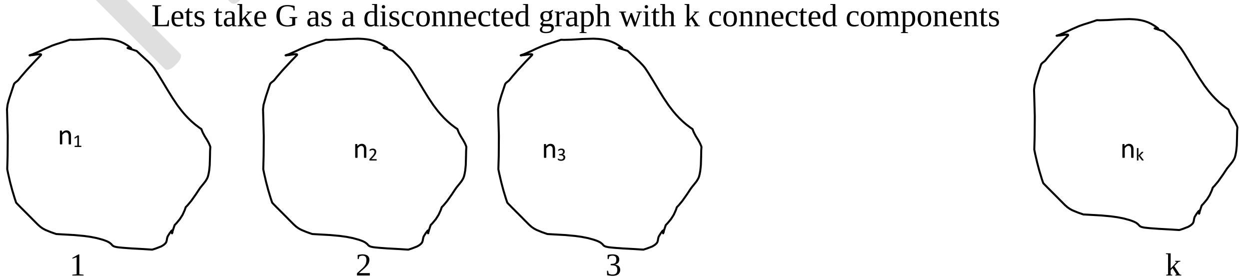
[The number of edges in complete graph with x vertices = ${}^{x}C_2$]

The number of edges cannot exceed ${}^{n1}C_2 + {}^{n2}C_2 + \ldots\ldots + {}^{nk}C_2$. If it exceeds then the number of connected components would be less than k.

7. The number of edges in a graph is more than (n-1) (n-2)/2 then graph is connected prove it.

Solution: we prove that every disconnected graph has $\leq$ (n-1) (n-2) /2 edges.

Lets take G as a disconnected graph with k connected components



Maximum number of edges it would have when every connected component is

Complete graph.

Let us assume that the number of vertices in a connected component "i" is n_i.

Then maximum number of edges any such graph can have =

$= {}^{n1}C_2 + {}^{n2}C_2 + {}^{n3}C_2 + \ldots\ldots\ldots + {}^{nk}C_2$

$= n_1 (n_1-1) /2 + n_2(n_2-1)/2 + n_3(n_3-1)/2 + \ldots\ldots\ldots + n_k(n_{k-1})/2.$

$= n_1^2 - n_1 /2 + n_2^2 - n_2/2 + n_3^2 - n_3/2 + \ldots\ldots\ldots + n_k^2 - n_k/2$

$= (n_1^2 + n_2^2 + n_3^2 + \ldots\ldots\ldots + n_k^2 - (n_1+n_2+n_3+\ldots\ldots\ldots+n_k))/2$ → eq1

Assume that graph has n vertices then

$n_1 + n_2 + n_3 + \ldots\ldots\ldots + n_k = n$

⇒ $(n_1 + n_2 + n_3 + \ldots\ldots\ldots + n_k)^2 = n_1^2 + n_2^2 + n_3^2 + \ldots\ldots\ldots + n_k^2 + 2n_1n_2 + 2n_2n_3 + \ldots\ldots\ldots 2n_{k-1} n_k$

⇒ $n_1^2 + n_2^2 + n_3^2 + \ldots\ldots n^2_k = (n_1 + n_2 + n_3 + \ldots\ldots + n_k)^2 - 2n_1n_2 - 2n_2n_3 - \ldots\ldots - 2n_{k-1} n_k$ → eq2

substitute eq2 in eq1

⇒ $((n_1 + n_2 + n_3 + \ldots\ldots n_k)^2 - 2n_1n_2 - 2n_2n_3 \ldots\ldots -2n_{k-1} n_k - (n_1 + n_2 + \ldots\ldots + n_k))/2$

Maximum number of edges = $(n^2 - 2(n_1n_2 + n_2n_3 + \ldots\ldots n_{k-1}n_k) - n)/ 2$ [since $n_1 + n_2 + \ldots\ldots n_k = n$]

⇒ $(N_2-n)/2 - [n1n2 + n_2n_3 + \ldots\ldots + n_{k-1} n_k\}$ → eq3

We have to prove that this value eq3 > (n-1) (n-2)/2

⇒ $n^2 - n /2 - (n_1n_2 + n_2n_3 + \ldots\ldots + n_{k-1} n_k) > n/n *((n-1) (n-2))/2$

⇒ $n^2 - n /2 - (n_1n_2 + n_2n_3 + \ldots\ldots + n_{k-1} n_k) > n (n-1)/2 [n-2/n]$

⇒ $n^2 - n /2 - (n_1n_2 + n_2n_3 + \ldots\ldots + n_{k-1} n_k) > n (n-1)/2 [1-2/n]$

⇒ $n^2 - n /2 - (n_1n_2 + n_2n_3 + \ldots\ldots + n_{k-1} n_k) > n (n-1)/2 - (n-1)$

⇒ $n (n-1)/2 - [n_1n_2 + n_2n_3 + \ldots\ldots + n_{k-1}n_k] > n(n-1)/2 - (n-1)$

⇒ $[n_1n_2 + n_2n_3 + \ldots\ldots + n_{k-1}n_k] < (n-1)$

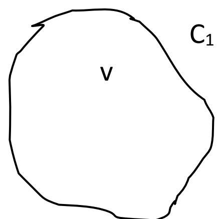
⟹ $n = n_1 + n_2 + \ldots\ldots + n_k$ then

⟹ $n-1 > n_1n_2 + n_2n_3 + \ldots\ldots n_{k-1} n_k$ is always false

8. If a graph has v1 vertices, each has degree at least (n-1) /2 then it is connected.

   Solution: Assume that graph G is not connected then it would have more than one connected component.

   C_1, C_2 are two connected components



   Let's assume that vertex v $\in C_1$

   Degree (v) $\geq$ (n-1)/2

   Then neighbors of v are (n-1)/2

   Since each neighbor belongs to same partition. C_1 contains atleast (n-1)/2 +1

   Vertices i.e. $\geq$ (n+1)/2

   Now other partition would have n-(n+1)/2 = (n-1)//2 vertices.

   Now since C_2 contains only (n-1)/2 vertices then neighbours of any vertex $\in C_2$ are

   (n-1)/2 which violates the rule that every vertex degree $\geq$ (n-1)/2

   Hence it is contradiction.

   That means we cannot have more than one component.

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❖ Practice questions:

1. A graph G(V, E) is constructed on set of integers and an edge is defined between two vertices iff the difference is 3 . The number of connected components in the graph

   a) 3 b) 2 C)4 d) 1

2. G1 (N, E1) and G2(N, E2) are two graphs on set of natural numbers . E1 is set of edges such that an edge is defined between two vertices iff the difference between vertices is 3. E2 is set of

edges such that an edge is defined between 2 vertices iff the difference between vertices is 4.The number of connected components in G1 U G2 = (W, E1 U E2) is

a) 1 b) 0 c)12 d)7

3. Edge connectivity number of K4 is

a) 1 b)2 c)3 d)4

4. Vertex connectivity number of K4 is

a) 1 b) 2 C)3 d)4

5. Edge connectivity number of a star graph is

a) 1 b) 2 c)3 d)4

6. Let G(V,E) be a simple connected graph with |V| vertices and |E| edges then how many number of cycles at least G can have.

a) |E| - |V|
b) |E| + |V|
c) |E| - |V| + 1
d) |E| + |V| - 1

7. Let G(V, E) a connected graph with u as cut- vertex then $\overline{G}$ – u is

a) Connected but does not have cycle
b) Disconnected
c) Connected
d) Disconnected but may have cycle

8. Let G has n vertices. If $\overline{G}$ is connected graph then the maximum number of edges that G can have is

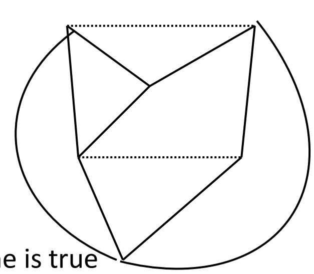
a) (n-1)(n-2)/2
b) n(n-1)/2
c) n-1
d) n

9. Let G be a graph with n vertices , where each vertex has degree at least n-2 (n≥ 5). Which of the following is true?

a) Cannot be bipartite
b) Cannot have cycle
c) Must be connected
d) Cannot have Euler path

10. Vertex connectivity and edge connectivity of the following graph

a) 2 , 4
b) 3, 4
c) 4, 3
d) 3, 3



11. Consider the following statements and which one is true

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S1: every degree 1 vertex in a graph has a neighbor that is a cut – vertex.

S2: every simple connected graph (n ≥ 2) with n vertices contains at least 2 vertices having the same degree

a) Only S1 b) Only S2 c) S1 and S2 d) none

12. Consider the following statements. Which one is true

S1: Bipartite graph cannot have an odd length cycle

S2: any graph with only even cycles is bipartite graph

a) Only s1 b) only s2 c) s1 and s2 d)none

13. Let G (V, E) be a graph and let e ∈ E . There exit two different cycles c1 and c2 both containing e then which one of the following statements is always true

a) e can be bridge in G
b) G has a cycle that does not contain e
c) Graph is connected
d) Graph is disconnected

TREES

❖ Practice Questions:

1. Let T = (V, E) be a tree and let d(v) be the degree of a vertex then $\sum_{u\epsilon\, v}(2-d(x))$
   a) 0
   b) 2
   c) |V|
   d) |E|

Solution: $\sum_{u\epsilon\, v} 2-d(x)$ = (2 – d (v1)) + (2 – d (v2)) +…… (2 – d (vn))

Where n = number of vertices

(2 + 2 + …… n times) – (d(v1) + d(v2) + …… + d(vn)) → eq1

In a tree the number of edges = n-1

d (v1) + d (v2) + …… d (vn) = 2 * |E|

d (v1) + d (v2) +……… d (vn) = 2 * (n-1) → eq2

Substitute eq2 in eq1

2n – (2n – 2)

= 2.

2. Let T (V, E) be a tree with a vertex of degree 3. prove that it cannot have Euler path or Euler cycle (|v| ≥ 3)

Solution: tree has at least 2 vertices of degree 1. The number of odd – degree vertices are more than 2 then graph cannot have Euler path or cycle.

3. A forest with n vertices and k connected components each component is a tree would have
   a) n - k edges

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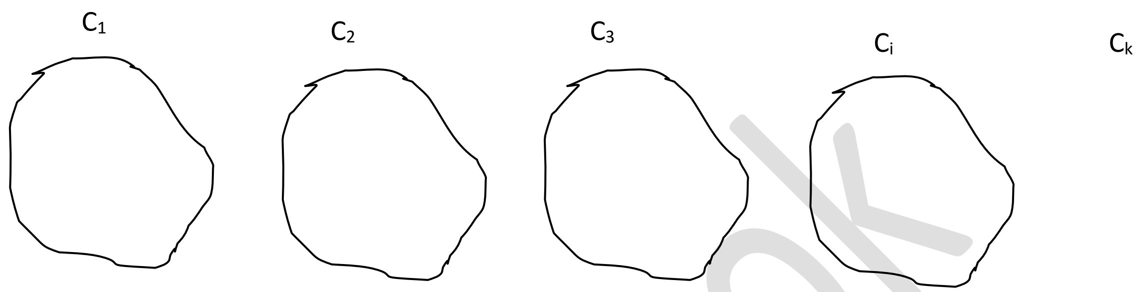
b) n – k + 1 edges

c) n + k - 2 edges

d) n + k edges

Solution:



C_i has $|C_i|$ vertices

The number of vertices in forest

= $|C_1| + |C_2| + \ldots\ldots\ldots + |C_k|$ → eq 1

Since each component is tree

The number of edges in component is $C_i = |C_i| - 1$ (tree property)

The total number of edges in a forest = $|C_1| -1 + |C_2| - 1 + \ldots\ldots |C_k| - 1$

$= \sum |C_i| - (1+ 1 +\ldots\ldots + k \text{ times})$

= n-k (from eq1)

4. Let G is a disconnected graph with (n-1) (n-2) /2 edges then $\overline{G}$ is a tree(n= number of vertices in G)

Solution: $|G| + |\overline{G}| = n (n-1) /2$

$(n-1)(n-2)/2 + x = n (n-1)/2$

$X= n (n-1) /2 – (n-1) (n-2)/2$

$X= (n-1)/2 [n-(n-2)]$

$X= n-1$

Since G is disconnected graph, $\overline{G}$ is connected. Connected graph with n-1 edges is always tree.

5. Let T be a tree with p vertices of degree 1 and q other vertices. Show that sum of the degrees of the vertices of degree greater than 1 is p + 2(q-1)

Solution: The number of vertices is p + q . so the number of edges is p +q -1

By sum of the degrees theorem

Total degree = $\sum d(v_i)$ = 1+ 1+ 1 +…… p times + $\sum_{\deg(vi)>1} \deg(vi)$

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$= p + \sum_{\deg(vi)>1} \deg(vi)$
$= 2 * e$
$= 2 * (p + q - 1)$ [e= no of vertices -1]

$P + \sum_{\deg(vi)>1} \deg(vi) = 2 * (p + q - 1)$
$\sum_{\deg(vi)>1} \deg(vi) = p + 2q - 2$
$\sum_{\deg(vi)>1} \deg(vi) = p + 2 * (q - 1)$

6. Show that if a tree has two vertices of degree 3 then it must have at least 4 vertices of degree 1

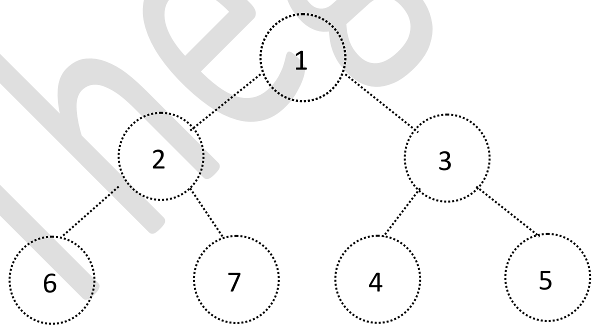
   Solution: Let v be the number of vertices. P be the number of vertices of degree 1 . If two vertices have degree 3, there are (v-p-2) vertices with degree other than 1 or 3 . In particular each of these (v- p – 2) vertices has degree at least 2.
   Sum of degrees = 2 * |E|
   Since (v - p - 2) vertices has degree at least 2
   $2 * (v - p - 2) + p * 1 + 3 * 2 \le 2 * |E|$
   $2v - 2p - 4 + p + 6 \le 2(v - 1)$
   $2v - p + 2 \le 2v - 2$
   $p \ge 4$

7. The number of internal nodes in a full m -ary tree is x then the number of nodes in tree is mx+ 1.

Solution: except root every node in a tree is child of only one parent or internal node. Each internal node has exactly m children
The number of children = m+ m+ m+……… + x times + 1 (root) = mx+1.

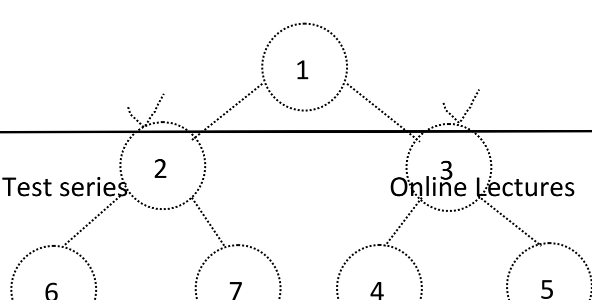
The number of nodes = number of children = mx + 1

Example:



Full 2-ary tree with 3 internal nodes.

Count the number of nodes like this.
To count children of each internal node, mark their children.
For example Mark children of 1.



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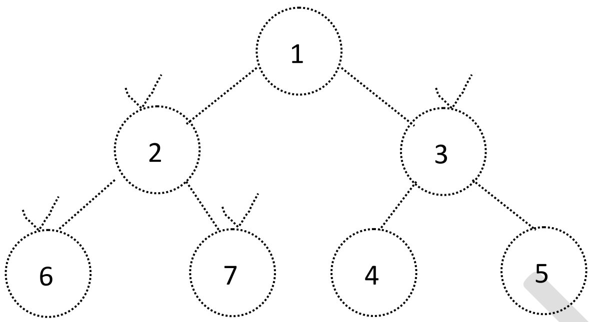
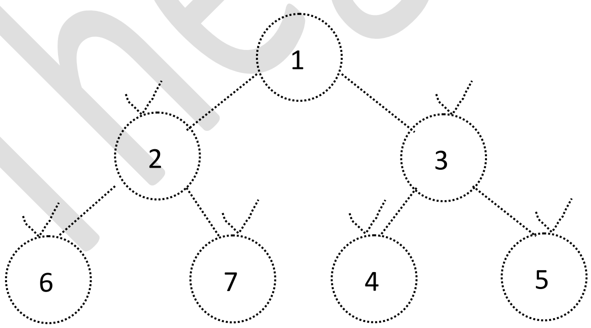
then mark children of 2



mark children of 3

Since every node is child node of exactly one internal node and each internal node has m children, the number of nodes = mn + root node = mn +1



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